Dissipative relativistic Magnetohydrodynamics of polarizable matter

David Wagner

Goethe-Universität Frankfurt

dwagner@th.physik.uni-frankfurt.de

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Outline

1. Reminder: Electrodynamics in matter
2. The macroscopic picture: Hydrodynamics
3. The microscopic picture: Kinetic Theory
4. Connecting the pictures
5. Results
6. Summary and Outlook

(Inhomogeneous) Maxwell Equations in matter

$$\partial_\mu F^{\mu\nu} = \mathcal{J}_f^\nu + \mathcal{J}_b^\nu \quad (1)$$

$$\partial_\mu H^{\mu\nu} = \mathcal{J}_f^\nu \quad (2)$$

$$\partial_\mu M^{\mu\nu} = -\mathcal{J}_b^\nu \quad (3)$$

where we have

- the **displacement tensor** $H^{\mu\nu} = F^{\mu\nu} + M^{\mu\nu} \equiv (D^\mu, H^\mu)$
- the **polarization tensor** $M^{\mu\nu} \equiv (-P^\mu, M^\mu)$
- the **fluid current** $\mathcal{J}_f^\mu = qN^\mu$
- the **bound current** $\mathcal{J}_b^\mu$

Electromagnetic Energy-Momentum Tensor

$$T_{em}^{\mu\nu} = \frac{1}{4} F^{\alpha\beta} F_{\alpha\beta} g^{\mu\nu} + H^{\mu}{}_{\alpha} F^{\alpha\nu} \qquad (4)$$

- There has been some debate over the form of this tensor $\rightarrow$ Abraham-Minkowski controversy (will not be covered here)
- We choose this form because it satisfies

$$\partial_\mu T_{em}^{\mu\nu} = -f^\nu = -\left(F^{\nu\rho} J_{f,\rho} + \frac{1}{2} M_{\alpha\beta} \partial^\nu F^{\alpha\beta} \right) \qquad (5)$$

where f^μ is the force density exerted on a piece of polarizable material

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Hydrodynamic equations

- All of hydrodynamics is based on conservation equations

Energy-momentum conservation

$$\partial_\mu T_f^{\mu\nu} = f^\nu \quad (6)$$

Particle four-current conservation

$$\partial_\mu N^\mu = 0 \quad (7)$$

- The question is now how to split these quantities meaningfully $\rightarrow$introduce **fluid 4-velocity** u^μ as a timelike four-vector

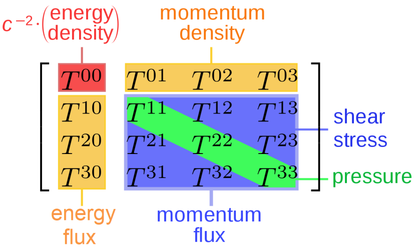
Decomposition of N^μ

- Utilize projection operators $u^\mu u^\nu$ and $\Delta^{\mu\nu} := g^{\mu\nu} - u^\mu u^\nu$

$$N^\mu = n u^\mu + n^\mu \tag{8}$$

  where
  - $n := u_\mu N^\mu$ constitutes the **particle number density**
  - $n^\mu := \Delta^\mu_\nu N^\nu$ is the **particle diffusion current**
- What to do with the Energy-momentum tensor?
- We can **not** assume it to be symmetric in the presence of polarizable matter...

Decomposition of $T_f^{\mu\nu}$



- We can decompose $T_f^{\mu\nu}$ as follows:

$$T_f^{\mu\nu} = \epsilon u^\mu u^\nu - (P_0 + \Pi)\,\Delta^{\mu\nu} + W^\mu u^\nu + \left(\tilde{W}_0^\nu + \tilde{W}^\nu\right) u^\mu + \pi^{\mu\nu} + \tilde{\pi}^{\mu\nu}$$

- Notice that we have to take care of antisymmetric parts

Decomposition of $T_f^{\mu\nu}$

Components of $T_f^{\mu\nu}$

- **energy density** $\epsilon := u_\mu u_\nu T_f^{\mu\nu}$
- **equilibrium and bulk viscous pressure** $P_0 + \Pi := \Delta_{\mu\nu} T_f^{\mu\nu}$
- **energy diffusion** $W^\mu := \Delta^\mu_\alpha u_\beta T_f^{\alpha\beta}$
- **momentum density** $\tilde{W}_0^\mu + \tilde{W}^\mu := \Delta^\mu_\beta u_\alpha T_f^{\alpha\beta}$
- **shear-stress tensor** $\pi^{\mu\nu} := \Delta^{\mu\nu}_{\alpha\beta} T_f^{\alpha\beta}$
- **antisymmetric momentum flux tensor**
  $\tilde{\pi}^{\mu\nu} := \frac{1}{2}\left(\Delta^\mu_\alpha \Delta^\nu_\beta - \Delta^\nu_\alpha \Delta^\mu_\beta\right) T_f^{\alpha\beta}$

$\Delta^{\mu\nu}_{\alpha\beta} := \frac{1}{2}\left(\Delta^\mu_\alpha \Delta^\nu_\beta + \Delta^\nu_\alpha \Delta^\mu_\beta\right) - \frac{1}{3}\Delta^{\mu\nu}\Delta_{\alpha\beta}$ denotes the traceless symmetric projector

Hydrodynamic quantities

- We are now dealing with **14 degrees of freedom**
  $\rightarrow n, n^\mu, \epsilon, \Pi, W^\mu, \pi^{\mu\nu}$
- Note that $\tilde{W}_0^\mu$, $\tilde{W}^\mu$ and $\tilde{\pi}^{\mu\nu}$ are no independent degrees of freedom
  $\rightarrow$ we will express them through the others later
- $\tilde{W}_0^\mu$ denotes the equilibrium, $\tilde{W}^\mu$ the dissipative part of the momentum density
- For the equilibrium quantities, we can immediately find equations of motion...

Equations of motion for ϵ, n, u^μ

EoM for equilibrium quantities

$$u_\alpha \partial_\beta T^{\alpha\beta} = -qE^\alpha n_\alpha + \frac{1}{2} M_{\alpha\beta} \frac{d}{d\tau} F^{\alpha\beta} \tag{9}$$

$$\Delta^\mu_\alpha \partial_\beta T^{\alpha\beta} = q\left(E^\mu n - Bb^{\mu\rho} n_\rho\right) + \frac{1}{2} M_{\alpha\beta} \nabla^\mu F^{\alpha\beta} \tag{10}$$

$$\partial_\mu N^\mu = 0 \tag{11}$$

with

$$\frac{d}{d\tau} := u^\mu \partial_\mu, \quad \nabla^\mu := \Delta^\mu_\nu \partial^\nu, \quad b^{\mu\nu} := \epsilon^{\mu\nu\alpha\beta} u_\alpha \frac{B_\beta}{B}, \quad B^2 := -B^\alpha B_\alpha$$

Viscous quantities

- Now we have exact equations for the equilibrium quantities
- What about the dissipative ones?
  $\Rightarrow$ Macroscopic conservation laws are not enough anymore, we have to refer to **kinetic theory**...

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Distribution function

- We now work with a **particle distribution function** which we split into an equilibrium and a dissipative part:

$$f = f_0 + \delta f$$

Equilibrium distribution function

$$f_0 = \left(e^{E_k \beta_0 + \alpha_0} + a \right)^{-1} \tag{12}$$

- β_0, α_0 denote the inverse temperature and chemical potential over temperature in equilibrium
- $E_{\mathbf{k}} := u_\mu k^\mu$ coincides with the energy in the particle rest frame
- a equals 1 for fermions, -1 for bosons and 0 for classical particles

Expansion of δf

- We expand δf (more specific: its momentum dependence) in a basis of irreducible tensors:

$$\delta f = \sum_{l=0}^{\infty} \alpha_{\mu_1 \cdots \mu_l} k^{\langle \mu_1} \cdots k^{\mu_l \rangle} \tag{13}$$

where

$$k^{\langle \mu_1} \cdots k^{\mu_l \rangle} = \Delta^{\mu_1 \cdots \mu_l}_{\nu_1 \cdots \nu_l} k^{\nu_1} \cdots k^{\nu_l}$$

denote the symmetric traceless projection

- These tensors form a **complete and orthogonal** set, analogously to the spherical harmonics

Reduction of the problem

- We have now reduced the problem to finding equations of motion for

$$\rho_r^{\mu_1\cdots\mu_n} := \langle E_{\mathbf{k}}^r k^{\langle\mu_1}\cdots k^{\mu_n\rangle}\rangle_\delta = \int \frac{d^3\mathbf{k}}{(2\pi)^3 k^0} E_{\mathbf{k}}^r k^{\langle\mu_1}\cdots k^{\mu_n\rangle}\delta f_{\mathbf{k}}$$

  and then truncate them at some order
- These **irreducible moments of δf** appear after expanding the coefficients in (13) as polynomials in energy
- In order to get the needed equations, we need to take a look at the **particle equations of motion** and the **Boltzmann Equation**

Kinetic theory for dipoles

- Since we are now operating on the level of individual particles, we need **Equations of Motion** for these

Mathisson-Papapetrou-Dixon (MPD) Equations

$$\frac{dk^\mu}{d\tau} = qF^\mu{}_\alpha \mathfrak{u}^\alpha + \frac{1}{2}\mathcal{M}_{\alpha\beta}\partial^\mu F^{\alpha\beta} \tag{14}$$

$$\frac{d\Sigma^{\mu\nu}}{d\tau} = 2k^{[\mu}\mathfrak{u}^{\nu]} - 2\mathcal{M}^{[\mu}{}_\alpha F^{\alpha\nu]} \tag{15}$$

$A^{[\mu}B^{\nu]} := A^\mu B^\nu - A^\nu B^\mu$

- $\mathcal{M}^{\mu\nu}$ denotes the **microscopic dipole tensor**, which entails the dipole moments of the particles
- $\Sigma^{\mu\nu}$ is the **spin tensor**

Kinetic theory for dipoles

- As a first approximation, we set

$$\frac{d}{d\tau}\Sigma^{\mu\nu} = 0$$

- This entails the assumption that the time scales of spin precession are much faster than the slowest microscopic time scale
  $\rightarrow$ this gives a relation between $\mathfrak{u}^{\mu}$ and k^{μ}
- In a weak-field approximation (neglecting terms of third order or higher in the electromagnetic fields) we have

$$\mathfrak{m}\mathfrak{u}^{\mu} \approx k^{\mu} + \frac{2}{\mathfrak{m}} k_{\alpha} F^{[\mu}{}_{\beta} \mathcal{M}^{\alpha]\beta} \equiv k^{\mu} + \pi^{\mu}$$

- The particle four-velocity is thus no longer parallel to the momentum, but is augmented by an **anomalous velocity**

Kinetic theory for dipoles

- A further novelty is the **modified mass shell**

Mass shell for dipoles

$$k^\mu k_\mu = m_0^2 + \frac{m_0}{2}\mathcal{M}_{\mu\nu}F^{\mu\nu} \qquad (16)$$

- (16) complicates life quite a bit, as now k^μ and u^μ do not share the same SO(3)-symmetry
  $\rightarrow$ No one-to-one correspondence of irreducible moments to hydrodynamic quantities

Kinetic theory for dipoles

- Electric and magnetic dipole moments can be determined from the MPD equations

Dipole moments

$$m_E^\mu = \kappa_E \mathcal{E}^\mu + \kappa' \mathcal{B}^\mu \quad (17)$$

$$m_B^\mu = \kappa_B \mathcal{B}^\mu + \kappa' \mathcal{E}^\mu \quad (18)$$

- $\kappa_E, \kappa_B, \kappa'$ are the electric, magnetic and mixed microscopic susceptibilities
- $\mathcal{E}, \mathcal{B}$ are the electric and magnetic fields in the **particle rest frame**

Kinetic theory for dipoles

Boltzmann equation

$$\delta \dot{f}_{\mathbf{k}} = -\dot{f}_0 + \frac{1}{E_{\mathbf{k}}}\left(1+\frac{1}{E_{\mathbf{k}}}u_\rho \pi^\rho\right) C[f] - \frac{1}{E_{\mathbf{k}}}\left[k^\rho\left(1+\frac{\pi^\mu u_\mu}{E_{\mathbf{k}}}\right) - \pi^\rho\right]\nabla_\rho f - \frac{m_0}{E_{\mathbf{k}}}\frac{dk^\mu}{d\tau}\frac{\partial}{\partial k^\mu} f \quad (19)$$

- $C[f]$ denotes the **collision kernel** which incorporates the underlying microscopic interactions

Kinetic theory for dipoles

- Utilizing the Boltzmann equation, we can find equations of motion for the ones of relevance
- In our case, that means finding expressions for

$$\dot{\rho}_r, \quad \dot{\rho}_r^{\langle\mu\rangle}, \quad \dot{\rho}_r^{\langle\mu\nu\rangle}$$

- These are quite lengthy due to the mass shell corrections and the anomalous velocity

Conserved quantities in Kinetic theory

- We can now express the conserved quantities from hydrodynamics in the language of kinetic theory (as collisional invariants)

Particle four-current

$$N^\mu = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3 k^0}\left[k^\mu f - m_0 \psi \frac{\partial f}{\partial k_\mu}\right] \tag{20}$$

Energy-Momentum Tensor

$$T_f^{\mu\nu} = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3 k^0}\left[\left(k^\mu k^\nu - m_0 \psi g^{\mu\nu}\right) f - m_0 \psi k^\nu \frac{\partial f}{\partial k^\mu}\right] \tag{21}$$

$\psi := -\frac{1}{4}\mathcal{M}_{\mu\nu}F^{\mu\nu}$ is the mass shell correction

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Connection between irreducible moments and hydrodynamics

- The modified mass shell lets the expressions for N^{μ} and $T_f^{\mu\nu}$ get quite lenghty
- Nevertheless, it is still possible to derive relations between the sought-after hydrodynamic quantities (as projections of $T_f^{\mu\nu}$ and N^{μ})
- This is the point where we have to make another **approximation**

14-moment approximation

- All moments $\rho_r^{\mu_1 \cdots \mu_l}$ with $l \geq 3$ are set to zero
- Also we reduce the space of **dynamical moments** (those that constitute degrees of freedom to

$$\rho_0, \ \rho_1, \ \rho_2, \ \rho_0^\mu, \ \rho_1^\mu, \ \rho_0^{\mu\nu}$$

- All other moments are expressed through these, meaning:

$$\begin{aligned} \rho_r &\propto \rho_0, \ \rho_1, \ \rho_2 \\ \rho_r^\mu &\propto \rho_0^\mu, \ \rho_1^\mu \\ \rho_r^{\mu\nu} &\propto \rho_0^{\mu\nu} \end{aligned}$$

- This corresponds to the leading order of an expansion in the **Knudsen number** $Kn = \frac{l_{micro}}{L_{macro}}$

- Now we have the hydrodynamic quantities as functions of the irreducible moments (and electromagnetic fields)
- Since we know the evolution of the latter, we also know the evolution of the former
- Writing this down requires a bit of work, since we first have to rediagonalize the system of equations, but is doable

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Equilibrium

- The Energy-Momentum Tensor is **not symmetric**, not even in equilibrium

Equilibrium Energy-Momentum Tensor

$$T^{\mu\nu}_{f,eq} = \epsilon u^\mu u^\nu - P_0 \Delta^{\mu\nu} + u^\mu \Pi^\nu \tag{22}$$

with

$$\mathbf{\Pi} = \mathbf{P} \times \mathbf{B} - \mathbf{M} \times \mathbf{E} \tag{23}$$

- This was also a result by Israel [1]
- So what's new?

Second-order relaxation equation

- Equilibrium dynamics is not all
- We also know the dynamics of dissipation (to second order in Kn and field strengths)
- Equations of motion for Π, n^μ, W^μ and $\pi^{\mu\nu}$ are very long; not shown here
- Two main effects:
  - known coefficients get corrections $\propto B^2 \propto \mathbf{M} \cdot \mathbf{B}$
  - new coefficients emerge, first and second order in Kn
- Now for some interesting limits

The massless spin 1/2 limit

- In the spin-1/2 case, the anomalous velocity π^μ vanishes
  $\rightarrow$ still lots of new terms
- This actually saves us a more complicated consideration, since massless particles do not feature SO(3) symmetry in velocity
- Also microscopic electric and mixed susceptibilities vanish

$$\Rightarrow \mathcal{M}^{\mu\nu} = \kappa_B \epsilon^{\mu\nu\alpha\beta} \mathfrak{u}_\alpha \mathcal{B}_\beta \tag{24}$$

- Reminder: Spins are assumed to be static
  $\rightarrow$ Assumption to be checked
- If we assume a Boltzmann gas with constant cross section, we can calculate coefficients analytically

The massless spin 1/2 limit

- Usually the bulk viscosity Π vanishes in the massless limit

Π in the massless limit

$$\Pi_{m_0=0} = \frac{13}{3}\tilde{\kappa}_B \epsilon^{\mu\nu\alpha\beta} W_\mu u_\nu E_\alpha B_\beta + \frac{8}{3}\tilde{\kappa}_B B^\mu B^\nu \pi_{\mu\nu} \qquad (25)$$

- The reason for this lies in the effective mass generated by dipole-field interactions

Navier-Stokes Limit

- We set all terms of order $\mathcal{O}(Kn^2)$ to zero and solve for dissipative quantities
- Acausal and unstable due to infinite signal propagation speed, but a good starting point for comparison nonetheless

Navier-Stokes limit (Eckart frame)

$$\Pi_{NS} = \zeta^{\mu\nu} \partial_\mu u_\nu \tag{26}$$

$$n^\mu_{NS} = \kappa^{\mu\nu}_\alpha \nabla_\nu \alpha_0 + \kappa^{\mu\nu}_\beta \nabla_\nu \beta_0 + \sigma^{\mu\nu}_E E_\nu \tag{27}$$

$$\pi^{\mu\nu}_{NS} = \eta^{\mu\nu\alpha\beta} \sigma_{\alpha\beta} - \eta_\theta \theta b^{\langle\mu} b^{\nu\rangle} - \eta_\omega b_\alpha \omega^{\alpha\langle\mu} b^{\nu\rangle} \tag{28}$$

$b^\mu := B^\mu / B, \qquad \theta = \Delta_{\mu\nu} \partial^\mu u^\nu, \qquad \sigma^{\mu\nu} = \Delta^{\mu\nu}_{\alpha\beta} \partial^\alpha u^\beta, \qquad \omega^{\mu\nu} = \nabla^{[\mu} u^{\nu]}$

Navier-Stokes Limit

Navier-Stokes limit (Eckart frame)

$$\Pi_{NS} = \zeta^{\mu\nu}\partial_\mu u_\nu \tag{29}$$

$$n^\mu_{NS} = \kappa^{\mu\nu}_\alpha \nabla_\nu \alpha_0 + \kappa^{\mu\nu}_\beta \nabla_\nu \beta_0 + \sigma^{\mu\nu}_E E_\nu \tag{30}$$

$$\pi^{\mu\nu}_{NS} = \eta^{\mu\nu\alpha\beta}\sigma_{\alpha\beta} - \eta_\theta \theta b^{\langle\mu} b^{\nu\rangle} - \eta_\omega b_\alpha \omega^{\alpha\langle\mu} b^{\nu\rangle} \tag{31}$$

- Can split all coefficients into components parallel/orthogonal to B^μ
- Coefficients are subject to B^2-corrections
- Dependencies (new ones in red):
  - $\Pi_{NS} \propto \theta, \sigma^{\mu\nu}$
  - $n^\mu_{NS} \propto \nabla^\mu \alpha_0, \nabla^\mu \beta_0, E^\mu$
  - $\pi^{\mu\nu} \propto \sigma^{\mu\nu}, \theta, \omega$

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Summary

- We have derived a theory of dissipative relativistic magnetohydrodynamics for polarizable and magnetizable fluids
- The dipole-field interaction complicates a lot of things, but the method of DNMR theory can still be applied (with some caveats)
- There are some quite restrictive assumptions in the derivation:
  - Restriction to weak fields
  - Static Spins
  - Classical description through kinetic theory
- However, the results are encouraging to undertake more research in this direction

Outlook

- Numerically simulate the resulting equations and compare with experiment
- Check stability and causality
- Relax the assumption of static spins
- Use quantum theory to get corrections for the derived classical results
- Take Landau quantization into account

Thank you for your attention!

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