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Kinetic theory of massive spin-1/2 particles from the Wigner-function formalism

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Motivation I

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- Early stages of non-central heavy-ion collisions: large orbital angular momentum and strong magnetic fields.
- Chiral magnetic effect (CME), chiral vortical effect (CVE): charge currents induced by magnetic and vortical fields.
- Similar effects for massive particles?
- Description tool: semiclassical kinetic theory.
- Question: how to derive kinetic theory and hydrodynamics from quantum field theory?
- For massive spin-0 particles, second-order dissipative magnetohydrodynamics has already been studied.
  G. Denicol, X-G Huang, E. Molnar, H. Niemi, J. Noronha, and D. H. Rischke, PRD98 (2018), 076009

Motivation II

- For massless particles, much work has been done already.
  J-Y. Chen, D. T. Son, and M. Stephanov, PRL 115 (2015), 021601;
  Y. Hidaka, S. Pu, D-L. Yang, PRD95 (2017), 091901;
  A. Huang, S. Shi, Y. Jiang, J. Liao, and P. Zhuang, arXiv:1801.03640 [hep-th]
- For massive particles, situation is more complicated: spin vector is additional degree of freedom, related to axial vector current.
- Covariant, next-to-leading order kinetic theory for massive spin-1/2 particles in inhomogeneous electromagnetic fields is still missing.
- Plan: use Wigner functions to derive kinetic theory.
- Similar studies have been made in equal time approach.
  Z. Wang, P. Zhuang et al., in preparation

Outline

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- Understand spin of relativistic massive particles.
- Understand classical limit.
- Understand massless limit.
- Transition of microscopic theory to macroscopic observables
  $\rightarrow$ Wigner functions.
- Analytically determine general Wigner function components.
  $\rightarrow$ Semi-classical expansion.
  $\rightarrow$ Comparison to massless case.
  Find generalized Boltzmann equation!
- Specify distribution function in global equilibrium.
- Determine hydrodynamic equations.

Classical Spin

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- Classical spin tensor $\Sigma_{\mu\nu}$ defined as intrinsic angular momentum about center of mass.
- Problem: center of mass of spinning particle is observer-dependent. $\rightarrow$ gauge freedom on $\Sigma_{\mu\nu}$.
- Let u^μ be four-velocity of an arbitrary frame. Then:

$$u^\nu \Sigma_{\mu\nu} = 0 \quad \Leftrightarrow \quad \Sigma_{\mu\nu} \text{ is intrinsic angular momentum about center of mass seen from frame with } u^\nu = (1, 0)$$

- Define spin tensor as intrinsic angular momentum tensor in the particle rest frame.

$$p_\nu \Sigma^{\mu\nu} = 0$$

Change of reference points

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- Intrinsic angular momentum tensors about centers of mass x_A and x_B are connected:

$$M_A^{\mu\nu} + x_A^\mu p^\nu - x_A^\nu p^\mu = M_B^{\mu\nu} + x_B^\mu p^\nu - x_B^\nu p^\mu$$

Conservation of total angular momentum.

- Pauli-Lubansky tensor: intrinsic angular momentum in lab frame
M. Stone, V. Dwivedi, and T. Zhou, PRD91 (2015), 025004

$$M_L^{\mu\nu} = \Sigma^{\mu\nu} - \Sigma^{\mu 0}\frac{p^\nu}{E} - \Sigma^{0\nu}\frac{p^\mu}{E},$$

$M_L^{i0} = 0$ by definition.

Spin in relativistic quantum theory I

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- Define spin tensor

$$\Sigma^{\mu\nu}_{rs} \equiv \frac{1}{m} \bar{u}(p,r)\sigma^{\mu\nu} u(p,s),$$

with spin operator $\sigma^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]$.

- From Dirac equation and adjoint:

$$p_\mu \Sigma^{\mu\nu}_{rs} = 0.$$

Semi-classically: gauge-fixed by Dirac equation.

- Spin vector:

$$n^\mu_{rs} = \frac{1}{2m} \bar{u}(p,r)\gamma^\mu \gamma^5 u(p,s)$$

Spin in relativistic quantum theory II

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- Choose spin quantization direction along polarization in local rest frame: $n^\mu_{rs} = sn^\mu \delta_{rs} = s(0, \vec{s})\delta_{rs}$ with polarization $\vec{s}$.
- Then $\Sigma^{\mu\nu}_{rs} = s\Sigma^{\mu\nu}\delta_{rs}$ with

$$\Sigma^{\mu\nu} = -\frac{1}{m}\epsilon^{\mu\nu\alpha\beta} p_\alpha n_\beta .$$

and in rest frame:

$$\begin{aligned} \Sigma^{ij} &= \epsilon^{ijk} n^k = \epsilon^{ijk} s^k , \\ \Sigma^{i0} &= 0. \end{aligned}$$

- Non-relativistic rotational properties hold in local rest frame!
- For massive particles: possible to Lorentz transform to different frame while keeping properties such as conservation of total angular momentum.

The massless spin tensor

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- Massless particles are different!
  State vectors transform under Lorentz transformations with additional phase.
- Remember: $\Sigma^{\mu\nu} = -\frac{1}{m}\epsilon^{\mu\nu\alpha\beta} p_\alpha n_\beta$.
- For massless particles, spin vector is parallel to momentum.
  $\rightarrow p^\mu \Sigma_{\mu\nu} = 0$ naturally satisfied, not a gauge condition.
- No local rest frame! No frame preferred.
- Define spin tensor in arbitrary frame u^μ
  J-Y. Chen, D. T. Son, and M. Stephanov, PRL 115 (2015), 021601

$$\Sigma_u^{\mu\nu} = -\frac{1}{p \cdot u}\epsilon^{\mu\nu\alpha\beta} u_\alpha p_\beta .$$

- Position has to be defined in same frame to conserve angular momentum. (A gauge-dependent position?)

Pauli-Lubansky tensor

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- Consider transformation between two frames in frame with $u^\mu = (1, 0)$

$$\Sigma_u^{\mu\nu} = \Sigma_{u'}^{\mu\nu} + \Sigma_{u'}^{\nu 0} \frac{p^\mu}{p^0} - \Sigma_{u'}^{\mu 0} \frac{p^\nu}{p^0} \equiv M_{L,u'} .$$

- For any choice of u' in spin tensor, corresponding Pauli-Lubansky tensor will by identical and equal to spin tensor in lab frame.
- Captures physical part of spin tensor!
- Need to define spin and position in observer's frame.
- Lorentz transformation of observer will change gauge.

Side-jump effect I

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J-Y. Chen, D. T. Son, and M. Stephanov, PRL 115 (2015), 021601;
M. Stone, V. Dwivedi, and T. Zhou PRL 114 (2015), 210402

- Collision of two right-handed massless particles, $p_1, p_2 \to p_3, p_4$.
- Center-of-mass frame: Ingoing and outgoing spins cancel (since momenta are parallel).
- Same situation as for spinless particles → "no-jump frame".
- Lorentz-boost observer to different frame A.
- Use Pauli-Lubansky tensor for conservation law in new frame, total ingoing angular momentum:

$$L^{\mu\nu}_{A,in} = \sum_{i=1,2} (x^\mu_{Ai} p^\nu_i - x^\nu_{Ai} p^\mu_i + M^{\mu\nu}_{Ai}).$$

- All quantities defined in frame A.

Side-jump effect II

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- Scattering amplitude depends on total angular momentum, this has to be observer-independent.

$$L^{\mu\nu}_{A,in} = L^{\mu\nu}_{CM,in} = 0 .$$

- From transformation between spin tensors in different frames, find shift between position as seen from CM and our observer:

$$x^\mu_{Ai} = x^\mu_{CMi} + \frac{1}{p^0_i (p_i \cdot u_{CM})} \epsilon^{\mu\nu\alpha 0} p_{i\alpha} u_{CM\nu} .$$

- Same holds for outgoing angular momentum with p_3, p_4.
- In collision: momentum changes. $\rightarrow$ Position changes.

What an observer will see

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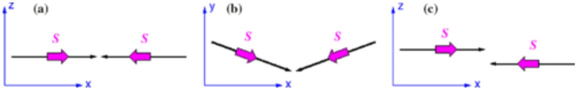
$$x^\mu_{Ai} = x^\mu_{CMi} + \frac{1}{p^0_i (p_i \cdot u_{CM})} \epsilon^{\mu\nu\alpha 0} p_{i\alpha} u_{CM\nu} .$$

- Center of mass: all worldlines pass through single collision point.
- Boosted parallel to momentum: after collision worldlines are shifted away from each other.
- Boosted perpendicular to momentum: already before collision worldlines are shifted away. Particles miss each other.



M. Stone, V. Dwivedi, and T. Zhou PRL 114 (2015), 210402

Wigner functions

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- Quantum analogue of classical distribution function.
- Contains information about quantum state of system.
- Off-equilibrium: two-point function depends not only on relative coordinate s, but also on central coordinate X.
- Wigner transformation of two-point function
  H.-Th. Elze, M. Gyulassy, and D. Vasak, AP 173 (1987)

$$W(X,p) = \int \frac{d^4 s}{(2\pi)^4} e^{-\frac{i}{\hbar} p\cdot s} \langle : \bar{\Psi}(X+\frac{s}{2}) U(X+\frac{s}{2}, X) U(X, X-\frac{s}{2}) \Psi(X-\frac{s}{2}) : \rangle$$

with gauge link

$$U(b,a) \equiv P \exp\left(-\frac{i}{\hbar}\int_a^b dz^\mu \, A_\mu(z)\right)$$

to ensure gauge invariance.

Transport equation

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- From Dirac equation: transport equation for Wigner function:
  H.-Th. Elze, M. Gyulassy, and D. Vasak, AP 173 (1987)

$$(\gamma_\mu K^\mu - m) W(X,p) = 0$$

with

$$\begin{aligned} K^\mu &\equiv p_W + \frac{1}{2} i\hbar \nabla^\mu , \\ \nabla^\mu &\equiv \partial_x^\mu - j_0(\Delta) F^{\mu\nu} \partial_{p\nu} , \\ p_W &\equiv p^\mu - \hbar \frac{1}{2} j_1(\Delta) F^{\mu\nu} \partial_{p\nu} , \end{aligned}$$

$\Delta = \frac{1}{2}\hbar\partial_p \cdot \partial_x$ with ∂_x only acting on $F^{\mu\nu}$ and $j_0(x) = \sin(x)/x$, $j_1(x) = [\sin(x) - x\cos(x)]/x^2$ spherical Bessel functions.

- Exact quantum kinetic equation for Wigner function for massive spin 1/2-particles and inhomogeneous fields!
- Only assumption: vanishing collision kernel, external classical gauge fields.

Strategy

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- Decompose W into generators of Clifford algebra.

$$W = \frac{1}{4}\left(\mathcal{F} + i\gamma^5\mathcal{P} + \gamma^\mu\mathcal{V}_\mu + \gamma^5\gamma^\mu\mathcal{A}_\mu + \frac{1}{2}\sigma^{\mu\nu}\mathcal{S}_{\mu\nu}\right).$$

- Insert into transport equation.
- Get system of 32 coupled (differential) equations.
- Equations for $\mathcal{F}$ (scalar, "distribution function") and $\mathcal{A}_\mu$ (axial-vector, "polarization") decouple from rest.
- Solve by expanding in powers of $\hbar$, assuming that Wigner function gradients, em field strengths and em field gradients are sufficiently small.
- Determine $\mathcal{V}_\mu$ ("vector current"), $\mathcal{P}$, $\mathcal{S}_{\mu\nu}$ from $\mathcal{A}_\mu, \mathcal{F}$.

Zeroth-order Wigner function

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- To zeroth order:

$$(p_\mu \gamma^\mu - m) W(X,p) = 0 .$$

  Wigner function is on-shell!
- Result directly calculated from definition is physical.
- Gauge link can be ignored in classical limit (no uncertainty).
- Choose spin quantization direction along polarization such that distribution function f_{rs} becomes diagonal in r, s.
- To simplify notation: only write positive-energy part.

Solution to zeroth order

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- Direct calculation yields

$$
\begin{aligned}
\mathcal{F}^{(0)} &= m\delta(p^2 - m^2)V \\
\mathcal{A}_\mu^{(0)} &= mn_\mu\delta(p^2 - m^2)A \\
\mathcal{P}^{(0)} &= 0 \\
\mathcal{V}_\mu^{(0)} &= p_\mu\delta(p^2 - m^2)V \\
\mathcal{S}_{\mu\nu}^{(0)} &= m\Sigma_{\mu\nu}\delta(p^2 - m^2)A,
\end{aligned}
$$

with

$$
\begin{aligned}
V &\equiv \frac{2}{(2\pi)^3}\sum_s f_s(x,p) \\
A &\equiv \frac{2}{(2\pi)^3}\sum_s s f_s(x,p)
\end{aligned}
$$

- Solution fulfills zeroth-order transport equation.

Next-to-leading order

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- To first order, Wigner function is no longer on-shell!
- Momentum variable of directly calculated Wigner function is not equal to physical momentum of particle $\rightarrow$ useless!
- Use transport equation!
- Insert zeroth-order solution into first-order equations.
- Find solution with correct massless limit.

Determine $\mathcal{F}$ and $\mathcal{A}^\mu$ up to order $\hbar$

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- Generalized Boltzmann equation:

$$p \cdot \nabla \mathcal{F} = \hbar \frac{1}{2} \partial_x^\lambda F^{\nu\rho} (\partial_{p\lambda} \mathcal{S}_{\nu\rho} + \partial_{p\rho} \mathcal{S}_{\nu\lambda}).$$

- Generalized spin transport equation:

$$p \cdot \nabla \mathcal{A}^\rho = F^{\rho\nu} \mathcal{A}_\nu + \hbar \frac{1}{6} \epsilon^{\mu\nu\lambda\rho} [(\partial_x^\alpha F_{\mu\lambda}) \partial_{p\alpha} \mathcal{V}_\nu + (\partial_{x\lambda} F_{\mu\sigma}) \partial_p^\sigma \mathcal{V}_\nu].$$

- Generalized on-shell conditions:

$$\begin{aligned} (p^2 - m^2) \mathcal{F} &= \frac{1}{2} \hbar F^{\mu\nu} \mathcal{S}_{\mu\nu}, \\ (p^2 - m^2) \mathcal{A}_\mu &= -\hbar \tilde{F}_{\mu\sigma} \mathcal{V}^\sigma. \end{aligned}$$

Determine $\mathcal{V}^\mu$, $\mathcal{P}$, and $\mathcal{S}^{\mu\nu}$ up to order $\hbar$

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- Only couple to $\mathcal{F}$ and $\mathcal{A}_\mu$:

$$\begin{aligned} \mathcal{V}^\mu &= \frac{1}{m}(p^\mu \mathcal{F} - \frac{1}{2}\hbar \nabla_\nu \mathcal{S}^{\nu\mu}), \\ \mathcal{S}^{\mu\nu} &= -\frac{1}{m}\epsilon^{\mu\nu\alpha\beta} p_\alpha \mathcal{A}_\beta + \hbar \frac{1}{2m}(\nabla^\mu \mathcal{V}^\nu - \nabla^\nu \mathcal{V}^\mu), \\ \mathcal{P} &= -\frac{1}{2m}\hbar \nabla_\mu \mathcal{A}^\mu . \end{aligned}$$

  Obviously only valid for $m \neq 0$!
- Can we still obtain massless limit?

Solution in next-to-leading order for $m = 0$

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- Equations decouple for $m = 0$!

$$\frac{1}{2}(\nabla_\mu J^\pm_\nu - \nabla_\nu J^\pm_\mu) = \pm\epsilon_{\mu\nu\alpha\beta} p^\alpha J^\beta_\pm$$

  for right- and left-handed currents $J^\pm_\mu \equiv \frac{1}{2}(\mathcal{V}_\mu \pm \mathcal{A}_\mu)$.

- Solution:
  Y. Hidaka, S. Pu, D-L. Yang, PRD95 (2017), 091901;
  A. Huang, S. Shi, Y. Jiang, J. Liao, and P. Zhuang, arXiv:1801.03640 [hep-th]

$$J^\pm_\mu = \left[p_\mu \delta(p^2) \pm \frac{1}{2}\hbar\epsilon_{\mu\nu\alpha\beta} p^\nu F^{\alpha\beta}\delta'(p^2) \pm \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}\frac{p^\alpha u^\beta}{p \cdot u}\delta(p^2)\nabla^\nu\right] f_\pm .$$

- u_β: four-velocity of an arbitrary frame.
  Remember expression for massless spin tensor!

Solution in next-to-leading order for $m \neq 0$

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- Find solutions for massive case:
  Replace massless by massive spin tensor in $\mathcal{A}_\mu$.

$$\begin{aligned}
\mathcal{F}^{(1)} &= m\delta(p^2 - m^2)V + \frac{\hbar}{2}\epsilon^{\mu\nu\alpha\beta}p_\mu n_\nu F_{\alpha\beta}\delta'(p^2 - m^2)A \\
\mathcal{A}^{\mu(1)} &= mn^\mu\delta(p^2 - m^2)A + \hbar\tilde{F}^{\mu\nu}p_\nu\delta'(p^2 - m^2)V \\
&\quad + \hbar\epsilon^{\mu\nu\alpha\beta}\frac{n^\alpha p^\beta}{2m}\delta(p^2 - m^2)\nabla_\nu V .
\end{aligned}$$

- Fulfill constraint equations!
- Found solution of transport equation with correct massless limit.

Vector current

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$$\mathcal{V}^{\mu(1)} = p^\mu \delta(p^2 - m^2) V + m\hbar \tilde{F}^{\mu\nu} n_\nu \delta'(p^2 - m^2) A$$
$$+\hbar \epsilon^{\mu\nu\alpha\beta} \frac{n_\alpha p_\beta}{2m} \delta(p^2 - m^2) \nabla_\nu A + \frac{1}{2m} \hbar \delta(p^2 - m^2) A \epsilon^{\nu\mu\alpha\beta} p_\alpha \nabla_\nu n_\beta .$$

- Convective part.
- Off-shell part $\leftrightarrow$ CME.
- Polarization part $\leftrightarrow$ CVE.
- Current is not parallel to momentum to first order!
- Parts orthogonal to momentum present for imbalance between spin-up and spin-down particles.

Generalized Boltzmann equation

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- Taylor expansion:

$$\delta(p^2 - m^2 - \hbar\frac{s}{2}F^{\mu\nu}\Sigma_{\mu\nu}) = \delta(p^2 - m^2) - \hbar\frac{s}{2}F^{\mu\nu}\Sigma_{\mu\nu}\delta'(p^2 - m^2) + O(\hbar^2),$$

- After some calculation:

$$\sum_s \delta(p^2 - m^2 - \frac{s}{2}\hbar F^{\alpha\beta}\Sigma_{\alpha\beta})\left\{p^\mu \partial_{x\mu} f_s + \partial_{p\mu}\left[F^{\mu\nu}p_\nu + \hbar\frac{1}{4}s\Sigma^{\nu\rho}(\partial^\mu F_{\nu\rho})\right] f_s\right\} = 0.$$

- Modified on-shell condition!
- Recover first Mathisson-Papapetrou-Dixon equation!
  W. Israel, General Relativity and Gravitation, vol. 9, no. 5 (1978), 451-468

Time evolution of spin

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- From kinetic equation for polarization to zeroth order:

$$m\frac{d}{d\tau}n^{\mu} = F^{\mu\nu}n_{\nu},$$

where τ is a worldline parameter with $\frac{d}{d\tau} = \dot{x^{\mu}}\frac{\partial}{\partial x^{\mu}} + \dot{p^{\mu}}\frac{\partial}{\partial p^{\mu}}$, where $\dot{x} \equiv \frac{\partial x}{\partial \tau}$.

- Recover BMT equation!
  V. Bargmann, L. Michel, and V. L. Telegdi, PRL 2 (1959)
- After some calculation:

$$m\frac{d}{d\tau}\Sigma^{\mu\nu} = \Sigma^{\lambda\nu}F^{\mu}_{\ \lambda} - \Sigma^{\lambda\mu}F^{\nu}_{\ \lambda}.$$

- Recover second Mathisson-Papapetrou-Dixon equation!
  W. Israel, General Relativity and Gravitation, vol. 9, no. 5 (1978), 451-468

Global equilibrium I

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- Up to now: completely generic distribution function.
  Now specify in simplest case: global equilibrium.
- Equilibrium distribution function:

$$f_s^{eq} = \left(e^{g_s} + 1\right)^{-1}$$

  with g linear combination of conserved quantities charge, momentum, and angular momentum:

$$g_s = \beta\pi \cdot U + \beta\mu_s - \frac{\hbar}{2} s \Sigma^{\mu\nu} \partial_\mu (\beta U_\nu).$$

  Here, $\pi_\mu \equiv p_\mu + A_\mu$ is canonical momentum, U is fluid velocity, $\beta \equiv \frac{1}{T}$ is inverse temperature, and μ_s is chemical potential.
- To zeroth order

$$f_s^{(0)} = \left(e^{g_0} + 1\right)^{-1}$$

  with

$$g_{s0} = \beta(\pi \cdot U - \mu_s).$$

Global equilibrium II

- "Homogeneous" part of the Boltzmann equation fulfilled if:

$$\begin{aligned} \mu_s &= \text{const}, \\ \partial_\nu \beta_\mu + \partial_\mu \beta_\nu &= 0, \\ \mathcal{L}_\beta F_{\mu\nu} &= 0. \end{aligned}$$

- "Inhomogeneous" part of Boltzmann equation: additional conditions to make global equilibrium possible.
- By Taylor expansion of distribution function:

$$\begin{aligned} V^{(1)\mu} = \frac{2}{(2\pi)^3} \sum_s \Big[& \delta(p^2 - m^2)(p^\mu + \hbar \frac{m}{2} s \tilde{\omega}^{\mu\nu} n_\nu \partial_{\beta\pi\cdot U}) \\ & + \hbar s \tilde{F}^{\mu\nu} n_\nu \delta'(p^2 - m^2) + s \frac{1}{2m} \hbar \delta(p^2 - m^2) \epsilon^{\nu\mu\alpha\beta} p_\alpha \nabla_\nu n_\beta \Big] f_s^{(0)}. \end{aligned}$$

- Thermal vorticity tensor: $\omega_{\mu\nu} \equiv \partial_\mu \beta_\nu - \partial_\nu \beta_\mu$.
- Dual thermal vorticity tensor: $\tilde{\omega}_{\mu\nu} \equiv \frac{1}{2} \epsilon_{\mu\nu\alpha\beta} \omega^{\alpha\beta}$.

Macroscopic quantities I

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- From Dirac Lagrangian in electro-magnetic field

$$\mathcal{L} = \bar{\Psi}\left[\frac{1}{2} i\hbar\gamma^\mu (D_\mu - D_\mu^\dagger) - m\right]\Psi$$

  we obtain:

- Number current:

$$\begin{aligned} J^\mu(x) &\equiv \langle : \bar{\Psi}(x)\gamma^\mu \Psi(x) : \rangle \\ &= \int d^4p\, V^\mu(x,p). \end{aligned}$$

- Canonical energy-momentum tensor (gauge-invariant form):

$$\begin{aligned} T^{\mu\nu} &= \left\langle : \frac{\partial \mathcal{L}}{\partial(D_\mu \Psi)} D^\nu \Psi + D^\nu \Psi^\dagger \frac{\partial \mathcal{L}}{\partial(D_\mu \Psi^\dagger)} - g^{\mu\nu}\mathcal{L} : \right\rangle \\ &= \int d^4p\, p^\nu V^\mu. \end{aligned}$$

Macroscopic quantities II

- Spin current tensor:

$$S^{\lambda,\mu\nu}(x) \equiv \frac{1}{4}\int d^4p\, \mathrm{tr}(\{\sigma^{\mu\nu},\gamma^\lambda\}W(x,p)) = -\frac{1}{2}\epsilon^{\mu\nu\lambda\rho}\int d^4p\, A_\rho(x,p).$$

- Total angular momentum tensor:

$$J^{\lambda,\mu\nu} = x^\mu T^{\lambda\nu} - x^\nu T^{\lambda\mu} + \hbar S^{\lambda,\mu\nu}.$$

Conservation laws

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- Conservation laws to first order:

$$\begin{aligned} \nabla_\mu \mathcal{V}^\mu &= 0, \\ \frac{1}{2}\hbar\epsilon^{\mu\nu\alpha\beta}\nabla_\alpha \mathcal{A}_\beta &= p_W^\nu \mathcal{V}^\mu - p_W^\mu \mathcal{V}^\nu, \\ &\Longrightarrow \\ \partial_\mu J^\mu &= 0, \\ \partial_\mu T^{\mu\nu} &= F^{\nu\mu} J_\mu, \\ \hbar\partial_\lambda S^{\lambda,\mu\nu} &= T^{\nu\mu} - T^{\mu\nu}, \\ \partial_\lambda J^{\lambda,\mu\nu} &= x^\mu F^{\lambda\nu} J_\lambda - x^\nu F^{\lambda\mu} J_\lambda . \end{aligned}$$

- Expected form of conservation laws!
- Energy-momentum tensor is conserved in combination with Maxwell part.
- Spin is not conserved separately.
- For zero electromagnetic fields, energy and total angular momentum are conserved.

Conclusions

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- Found transport equation for distribution function and polarization for massive spin-1/2 particles in inhomogeneous electromagnetic fields.
- Recovered classical equations of motion.
- Found a way of obtaining massless limit.
- Showed agreement of our solution to previously known massless solution in this limit.
- Gave explicit expressions for current in global equilibrium.

Outlook

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- Generalized Boltzmann equation still has to be solved.
- Collisions have to be included.
  $\rightarrow$ Boltzmann equation without assumption of local equilibrium.
- Derive equations of motion for dissipative quantities.
  $\rightarrow$ Method of moments.