Entropy production in relativistic heavy ion collisions with use of quantum distribution functions

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Based on

PTEP, 083A01 (2015).
PRD 94, 091502(R) (2016).
PTEP, 013D02 (2017).

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  Relativistic heavy ion collisions
  Thermalization and entropy production

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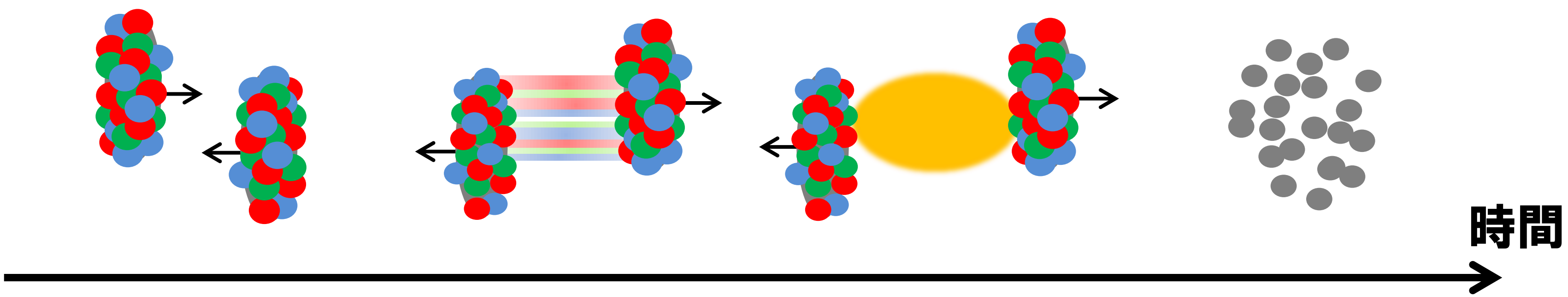
Relativistic heavy ion collisions



Color Glass Condensate (CGC)
[McLerran-Venugopalan (1993,1994,1994)]

Glasma
[Lappi-McLerran(2006)]

Quark Gluon Plasma (QGP)

Hadron

Hydro model

Early thermalization problem

[Heinz(2002)]

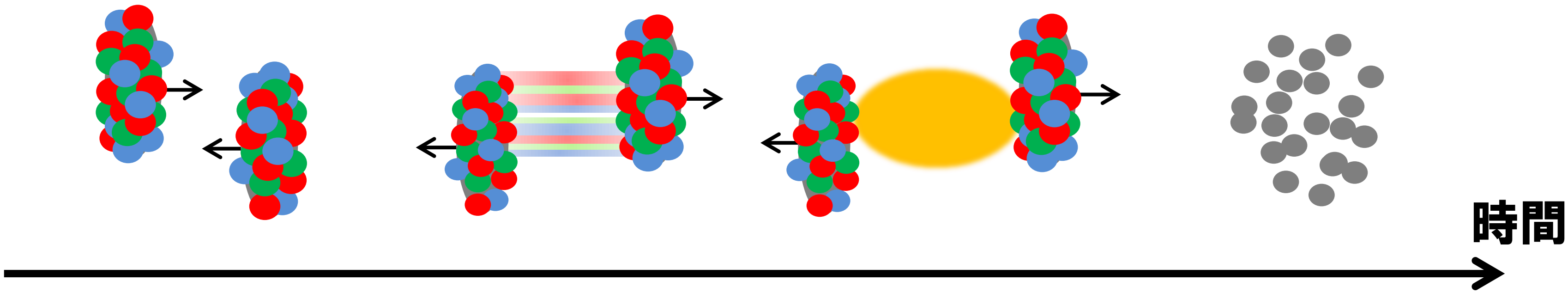
Initial time of hydro model ~0.6-1.0fm/c

(Review:[Kolb-Heinz(2003)])

pQCD cannot explain this time.

[Baier-Mueller-Schiff-Son(2001)]

Relativistic heavy ion collisions



Color Glass Condensate (CGC)
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Quark Gluon Plasma (QGP)

Hadron

Classical Yang-Mills
[Romatschke-Venugopalan(2006)]

Classical YM field
＋fluctuation
⇒Instability

Classical statistical approximation
[Epelbaum-Gelis(2013)]

Early isotropization

Hydro model

Early thermalization problem
[Heinz(2002)]

Initial time of hydro model ～0.6-1.0fm/c
(Review:[Kolb-Heinz(2003)])

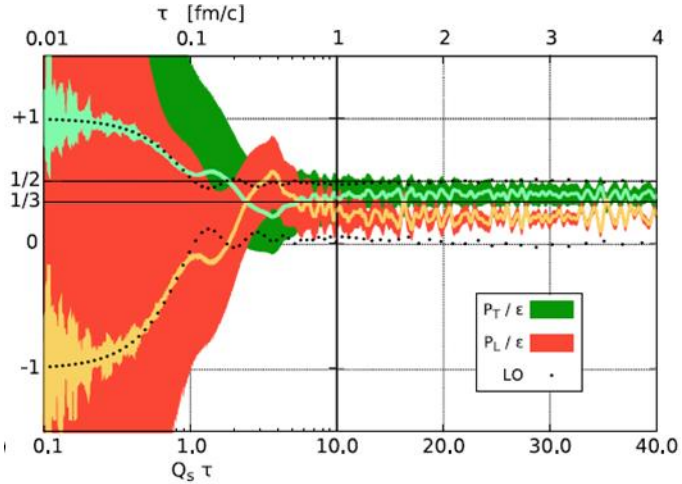
pQCD cannot explain this time.
[Baier-Mueller-Schiff-Son(2001)]

How to show "thermalization"?

(A) Isotropization of pressure: Setup of perfect fluid

[Arnold-Lenaghan-Moore-Yaffe(2004),Epelbaum-Gelis(2013)]

Energy-momentum tensor $T_{ij}^{(\mathrm{YM})} \simeq p\delta_{ij}$ Ideal hydro model



Isotropization of the pressure

(SU(2)YM on $64 \times 64 \times 128$ lattice)

[Epelbaum-Gelis(2013)]

How to show "thermalization"?

(A) Isotropization of pressure: Setup of perfect fluid

[Arnold-Lenaghan-Moore-Yaffe(2004),Epelbaum-Gelis(2013)]

Energy-momentum tensor $T_{ij}^{(\mathrm{YM})} \simeq p\delta_{ij}$ Ideal hydro model

(B) Justification of hydro model: Comparing with (viscus) fluid simulation

[Kurkela-Zhu(2015)] [Keegan-Kurkela-Mazeliauskas-Teaney(2016)]

(C) Local thermal equilibrium

We should show entropy production. But there are few previous works.

Phenomenological estimate $\Delta S/\Delta Y \simeq 4500$ [Muller-Schafer(2011)]

(per rapidity)

Theoretical calculation

1) Entropy production by decoherence, [Muller-Schafer(2005), Fries-Muller-Schafer(2008)] [Iida-Kunihiro-Ohnishi-Takahashi(2014)]

2) Entropy production rate expected by chaoticity Review: [Biro-Matinyan-Muller(1994)]

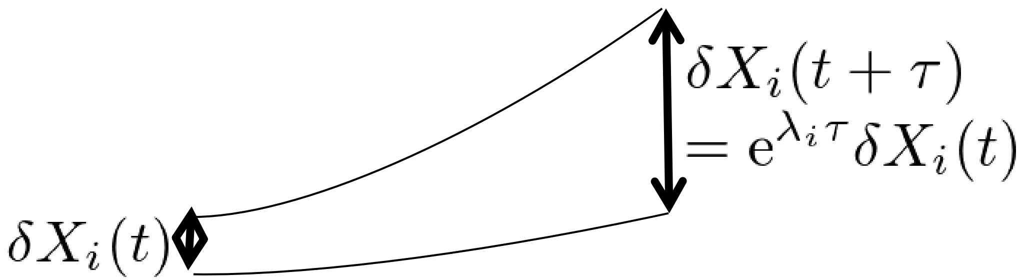
Entropy production suggested by chaoticity

Classical YM field is a chaotic system. [Matinyan-Savvidy-Ter-Arutunian-Savvidy(1981), Chirikov-Shepelyansky(1981),Nikolaevsky-Shchur(1982)]

Measure of chaoticity

Lyapunov exponents λ_i

$$|\delta X_i(t)| \propto \mathrm{e}^{\lambda_i t},$$



$\delta \vec{X}$:Distance between two points in phase space

In the case of regular systems, $\lambda_i = 0$.

The sum of positive Lyapunov exponents

= Kolmogorov-Sinai(KS) entropy (KS rate)

$$\lambda_{\mathrm{KS}} = \sum_{\lambda_i > 0} \lambda_i$$ [Pesin(1977)]

It characterizes the growth rate of physical entropy.

[Latora-Baranger (1999), Baranger-Latora-Rapisarda (2002)]

Lyapunov exponents in classical YM field

Due to the scale invariance, $\lambda_i(, \lambda_{\mathrm{KS}}) = c \times \varepsilon^{1/4}$ ε : energy density

Numerical calculation of Lyapunov exponents

Review: [Biro-Matinyan-Muller(1994)] [Kunihiro et al.(2010),Iida et al.(2013)]

Time evolution operator of $\delta\vec{X}$

$$U(t, t+\tau) = \mathcal{T}[\exp(\int_t^{t+\tau} \mathcal{H}(t')dt')]$$

$\mathcal{H}$: Hessian

τ is finite in numerical calculation.

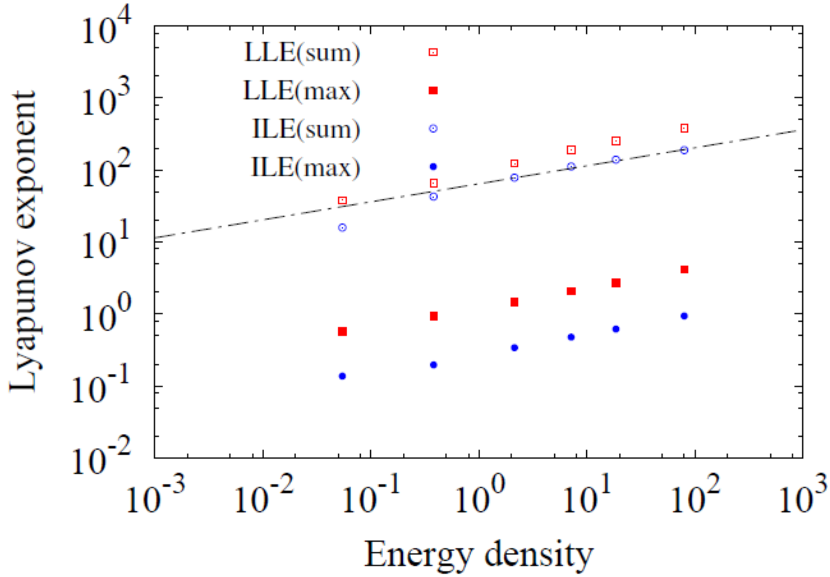
(a) Local Lyapunov exponent(LLE): τ is infinitesimal
It depends on the initial condition.

(b)Intermediate Lyapunov exponent(ILE): τ is finite
It characterizes chaoticity of the system.

$$\lambda_{\mathrm{KS}}^{\mathrm{LLE}}/L^3 = 1.9 \times \varepsilon^{1/4},$$
$$\lambda_{\mathrm{KS}}^{\mathrm{ILE}}/L^3 = 1.0 \times \varepsilon^{1/4}.$$

Lyapunov exponents in SU(2)YM

[H.T. et al.(2016)]



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Husimi-Wehrl entropy

It is defined in “phase space” in quantum mechanics.

- Wigner function : Weyl-Wigner transformation of density matrix [Wigner(1932)]

$$f_{\mathrm{W}}(q,p) = \int_{-\infty}^{\infty} d\eta \mathrm{e}^{-ip\eta/\hbar} \langle q + \frac{\eta}{2} | \hat{\rho} | q - \frac{\eta}{2} \rangle$$

It is a quasi-distribution function, because it is not semi-positive definite.

$$\langle \hat{A} \rangle = \mathrm{Tr}[\hat{A}\hat{\rho}] = \int dq dp A_{\mathrm{W}}(q,p) f_{\mathrm{W}}(q,p)$$

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- Husimi function : Semi-positive definite function [Husimi(1940)]

$$f_{\mathrm{H}}(q,p;t) = \int \frac{dq'dp'}{\pi\hbar} \exp(-\frac{\Delta}{\hbar}(q-q')^2 - \frac{1}{\Delta\hbar}(p-p')^2) f_{\mathrm{W}}(q',p';t)$$
$$= \langle \alpha | \hat{\rho} | \alpha \rangle = \sum_i \omega_i |\langle \alpha | \psi_i \rangle|^2 \geq 0.$$

$|\alpha\rangle$: coherent state

$$\hat{\rho} = \sum_i \omega_i |\psi_i\rangle\langle\psi_i|$$

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- Husimi-Wehrl (HW) entropy [Wehrl(1978)]

$$S_{\mathrm{HW}}(t) = -\int dq dp f_{\mathrm{H}}(q,p;t) \log f_{\mathrm{H}}(q,p;t)$$

The growth rate at large times in inverse harmonic oscillator agrees with a Lyapunov exponent.

The HW entropy shows similar behavior with von Neumann entropy in the case of a harmonic oscillator at finite temperature. [Kunihiro-Muller-Ohnishi-Schafer(2009)]

Difficulties in numerical calculation

1. A lot of degrees of freedom

Field theory has a lot of degrees of freedom(DOF).

2. Time evolution

The time evolution of the Husimi function is cumbersome. [Tsai-Muller(2010)]

Difficulties in numerical calculation

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⇨ Our numerical method is based on Monte-Carlo method, which is applicable in a system with large DOF.

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The time evolution of the Husimi function is cumbersome. [Tsai-Muller(2010)]

⇨ We calculate the time evolution of the Wigner function in the semiclassical approximation.

$$\frac{\partial}{\partial t} f_{\mathrm{W}}(q,p) = \frac{\partial H_{\mathrm{W}}}{\partial q}\frac{\partial f_{\mathrm{W}}}{\partial p} - \frac{\partial H_{\mathrm{W}}}{\partial p}\frac{\partial f_{\mathrm{W}}}{\partial q} + \mathcal{O}(\hbar^2)$$

This means that the Wigner function is constant along classical trajectories.

The approximation is valid in the early stage of the heavy ion collisions.

Test particle methods

H.T., H.Iida, T.Kunihiro, A.Ohnishi, and T.T.Takahashi, PTEP, 083A01 (2015).
H.T., H.Iida, T.Kunihiro, A.Ohnishi, and T.T.Takahashi, PRD 94, 091502(R) (2016).

Husimi-Wehrl entropy : Written by Wigner function explicitly

$$S_{HW}(t) = -\int \frac{d\vec{p}d\vec{q}}{(2\pi\hbar)^n} \exp(-\frac{1}{\Delta\hbar}\vec{p}^2 - \frac{\Delta}{\hbar}\vec{q}^2) \int \frac{d\vec{p'}d\vec{q'}}{(\pi\hbar)^n} f_W(\vec{p'},\vec{q'};t)$$
$$\times \log \int \frac{d\vec{p''}d\vec{q''}}{(\pi\hbar)^n} \exp(-\frac{1}{\Delta\hbar}(\vec{p}+\vec{p'}-\vec{p''})^2 - \frac{\Delta}{\hbar}(\vec{q}+\vec{q'}-\vec{q''})^2) f_W(\vec{p''},\vec{q''};t)$$

Test particle method

We assume that the Wigner function is written by the sum of delta functions.

$$f_W(\vec{p},\vec{q};t) = \frac{(2\pi\hbar)^n}{N} \sum_i^N \delta^{(n)}(\vec{p}-\vec{\bar{p}}^i(t))\delta^{(n)}(\vec{q}-\vec{\bar{q}}^i(t)))$$

According to the initial Wigner function, an ensemble of test particles is generated.

Each test particle obeys classical EOM.

$$\dot{q}_i = \frac{p_i}{m}, \dot{p}_i = -\frac{\partial V}{\partial q_i}$$

Test particle methods

H.T., H.Iida, T.Kunihiro, A.Ohnishi, and T.T.Takahashi, PTEP, 083A01 (2015).
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$$\times \log \int \frac{d\vec{p''}d\vec{q''}}{(\pi\hbar)^n} \exp(-\frac{1}{\Delta\hbar}(\vec{p}+\vec{p'}-\vec{p''})^2 - \frac{\Delta}{\hbar}(\vec{q}+\vec{q'}-\vec{q''})^2) f_W(\vec{p''},\vec{q''};t)$$

Sets of test particles

$\{\Gamma'_i\}$ $\{\Gamma''_i\}$

Test particle method

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According to the initial Wigner function, an ensemble of test particles is generated.

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$$\dot{q}_i = \frac{p_i}{m}, \dot{p}_i = -\frac{\partial V}{\partial q_i}$$

When we prepares same sets:

$$\{\Gamma'_i\} = \{\Gamma''_i\}$$

Test particle(TP) method

When we prepare different sets:

$$\{\Gamma'_i\} \neq \{\Gamma''_i\}$$

Parallel test particle(pTP) method

Entropy production in quantum mechanics

H.T., H.Iida, T.Kunihiro, A.Ohnishi, and T.T.Takahashi, PTEP, 083A01 (2015).
H.T., H.Iida, T.Kunihiro, A.Ohnishi, and T.T.Takahashi, PRD 94, 091502(R) (2016).

Hamiltonian

$$H = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} + \frac{1}{2}g^2 q_1^2 q_2^2 + \frac{\epsilon}{4}q_1^4 + \frac{\epsilon}{4}q_2^4$$

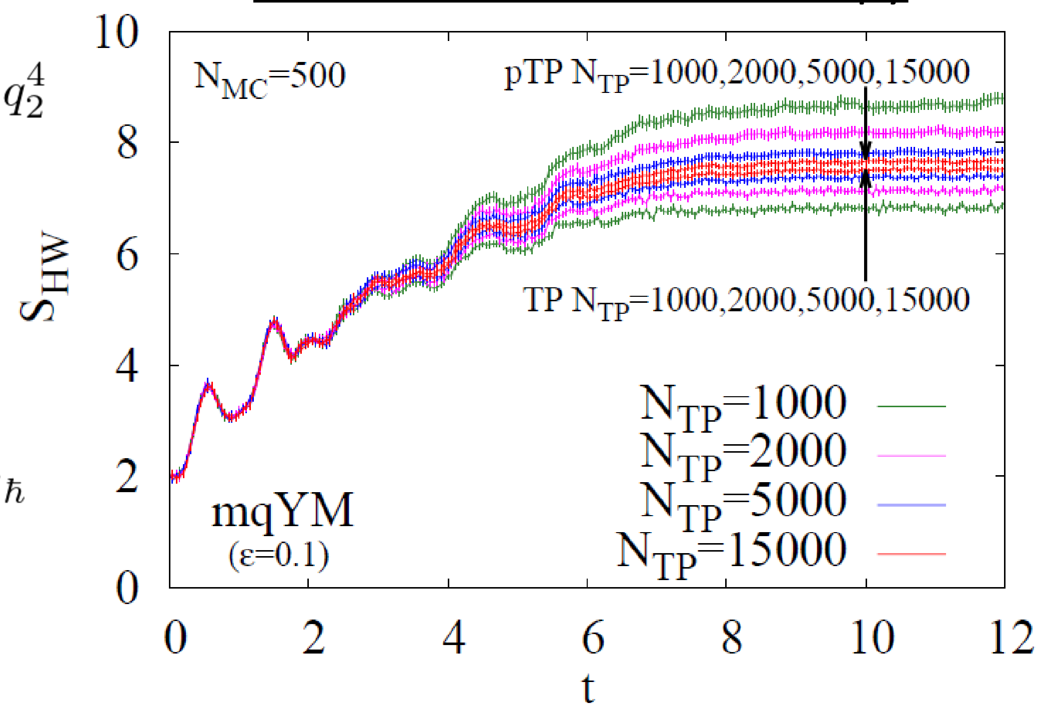
$$m = 1, g = 1, \epsilon = 0.1$$

Initial condition: coherent state

$$f_{\mathrm{W}}(\vec{p}, \vec{q}, t = 0)$$
$$= 4e^{-[q_1^2 + q_2^2 + (p_1 - 10)^2 + (p_2 - 10)^2]/\hbar}$$

Our methods can describe the entropy production.

Time evolution of HW entropy



With increasing the number of test particles, the HW entropy converges from below (above) in the TP (pTP) method.

Difficulties in numerical calculation

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Field theory has a lot of degrees of freedom.

⇨ Our numerical method is based on Monte-Carlo method, which is applicable in the system with large DOF.

2. Time evolution

The time evolution of the Husimi function is cumbersome. [Tsai-Muller(2010)]

⇨ We calculate the time evolution of the Wigner function in the semiclassical approximation.

The approximation is valid in the early stage of the heavy ion collisions.

3. Small overlap between f_H and $-\log f_H$

The importance sampling for the both $-\log f_H$ and f_H is difficult.

⇨ We use an additional approximation called "product ansatz" in a system with large DOF.

Calculation with product ansatz

H.T., H.Iida, T.Kunihiro, A.Ohnishi, and T.T.Takahashi, PRD 94, 091502(R) (2016).

Product ansatz

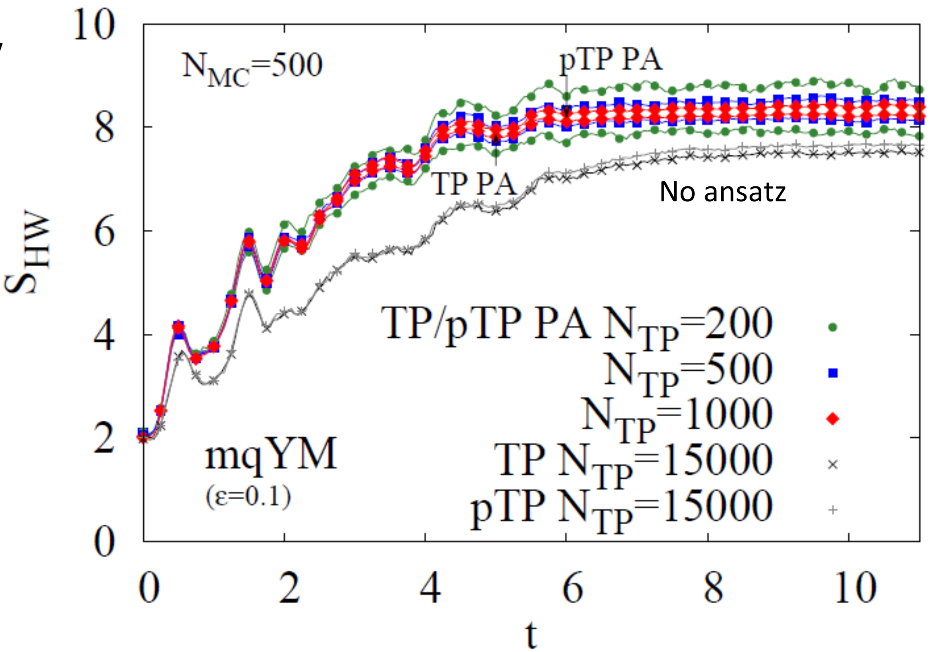
We assume Husimi function is written by the sum of Husimi function in 1 DOF.

$$f_{\mathrm{H}}(\vec{q},\vec{p};t) = \prod_i^D h_i(q_i,p_i;t)$$

$$S_{HW}^{(PA)} = -\sum_i^D \int \frac{dq_i dp_i}{2\pi\hbar} h(q_i,p_i;t)\log h(q_i,p_i;t)$$

$$\geq S_{HW}$$ ∵ Subadditivity of entropy

Time evolution of HW entropy



Product ansatz gives the upper bound of the entropy.
The HW entropy in product ansatz agrees with the results without the ansatz within 10-20% error.
It is efficient from the view point of the numerical cost reduction.

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Schrodinger picture in Yang-Mills theory

Hamiltonian in temporal gauge, $A_0^a = 0$

$$H = \frac{1}{2}\sum_{x,a,i} E_i^a(x)^2 + \frac{1}{4}\sum_{x,a,i,j} F_{ij}^a(x)^2$$

$$F_{ij}^a = \partial_i A_j^a(x) - \partial_j A_i^a(x) + g\sum_{b,c} f^{abc} A_i^b(x) A_j^c(x)$$

Canonical variables $(A_i^a(x), E_i^a(x))$

$$[\hat{E}_i^a(t,\mathbf{x}), \hat{A}_j^b(t,\mathbf{y})] = \delta^{ab}\delta_{ij}\delta^{(3)}(\mathbf{x}-\mathbf{y})i\hbar.$$

Wave functional [Tomonaga(1946)] [Symanzik(1981),Luscher(1985)][Cooper-Mottola(1987),Pi-Samiullah(1987)]

$$i\frac{\partial}{\partial t}|\Psi\rangle = H[\hat{A},\hat{E}]|\Psi\rangle,$$

T^a:generator of SU(N)

$A = A^a T^a, E = E^a T^a$

Condition of physical state (Gauss law)

$$[\hat{D}_i, \hat{E}_i]|\Psi\rangle = 0 \qquad \hat{D}_i = \partial_i - ig\hat{A}_i$$

Quantum distribution functional

- <u>Wigner functional</u> [Mrowczynski-Muller(1994)][BialynickiBirula(2000)] [Fukushima-Gelis-McLerran(2007)][Jeon-Epelbaum(2016)] $(A \to g^{-1}A, E \to g^{-1}E)$

Density matrix $\hat{\rho} = |\Psi(t)\rangle\langle\Psi(t)|$

$$f_{\mathrm{W}}[A,E;t] = \int \frac{DA'}{g}\, e^{iE\cdot A'/\hbar g^2}\, \langle A + A'/2 \mid \hat{\rho}(t) \mid A - A'/2\rangle$$

<u>Initial condition: Gaussian packet</u> (physical state)

$$\Psi[A;t=0] = \langle A|\Psi(t=0)\rangle = \mathcal{N}\exp\left[-\frac{\omega}{2\hbar g^2}(A-\bar{A})^2 - \frac{i}{\hbar g^2}\bar{E}\cdot(A-\bar{A})\right],$$

$$f_W[A,E;t=0] = 2^{N_D}\exp\left(-\frac{\omega(A-\bar{A})^2}{\hbar g^2} - \frac{(E-\bar{E})^2}{\omega\hbar g^2}\right),$$

- <u>Husimi functional & HW entropy</u>

$$f_{\mathrm{H}}[A,E;t] = \int \frac{DA'DE'}{(\pi\hbar g^2)^{N_{\mathrm{D}}}}\exp\left(-\frac{\Delta(A-A')^2}{\hbar g^2} - \frac{(E-E')^2}{\Delta\hbar g^2}\right) f_{\mathrm{W}}[A',E';t],$$

$$S_{\mathrm{HW}}(t) = -\int D\Gamma\, f_{\mathrm{H}}[A,E;t]\log f_{\mathrm{H}}[A,E;t]. \qquad D\Gamma = \frac{DADE}{(2\pi\hbar g^2)^{N_{\mathrm{D}}}}$$

The HW entropy is invariant under the residual gauge freedom of the temporal gauge.

Entropy production with random initial condition

H.T., H.Iida, T.Kunihiro, A.Ohnishi, and T.T.Takahashi, PRD 94, 091502(R) (2016).
H.T., T.Kunihiro, A.Ohnishi, and T.T.Takahashi, PTEP, 013D02 (2017).

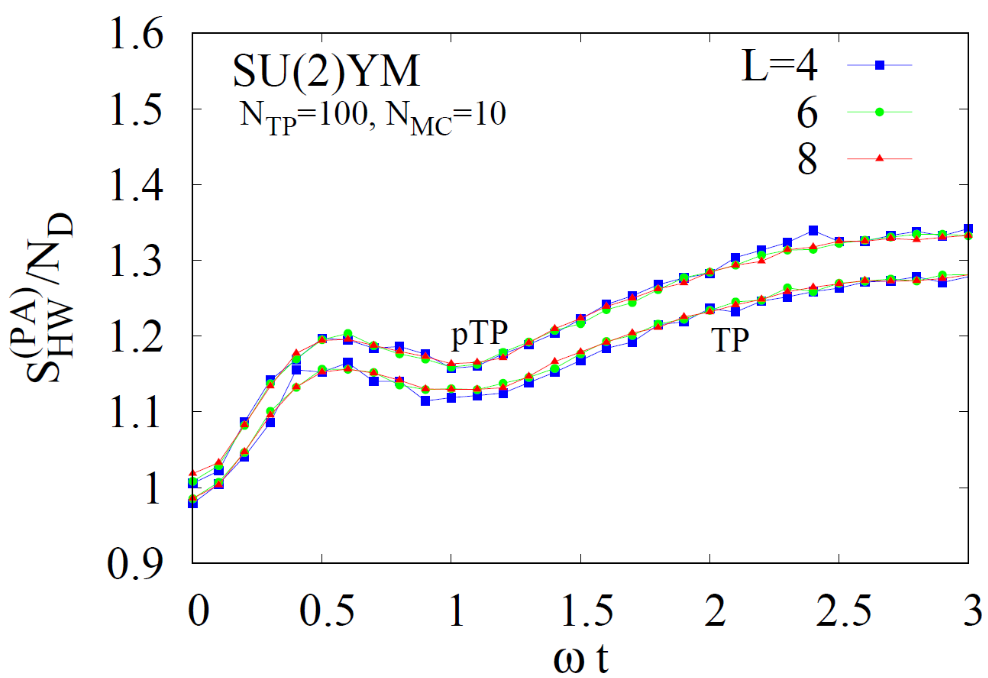
Initial condition : ($\bar{A}, \bar{E} = 0$)

$$f_{\mathrm{W}}[A, E; t = 0] = 2^{N_D} \exp(-\omega A^2/\hbar g^2 - E^2/\omega\hbar g^2)$$

The results do not have the lattice size dependence.

The numerical behavior is same as that of quantum mechanics.

Time evolution of HW entropy per DOF



By using the product ansatz, we can calculate the entropy production in Yang-Mills field theory.

Entropy production with random initial condition

H.T., H.Iida, T.Kunihiro, A.Ohnishi, and T.T.Takahashi, PRD 94, 091502(R) (2016).
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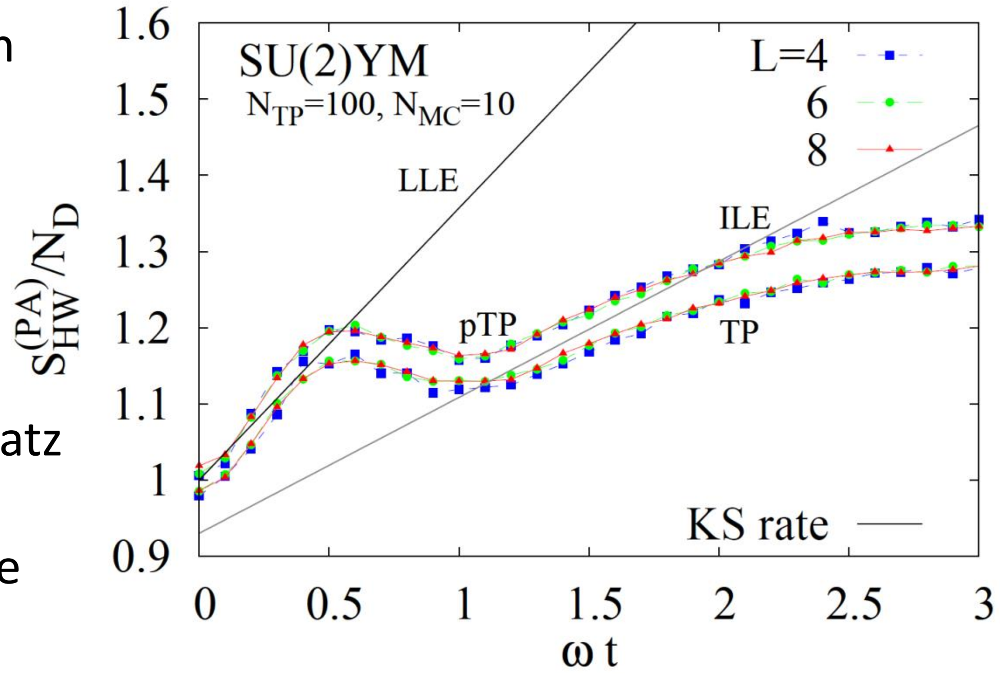
Comparing with Lyapunov exponents

The LLE (ILE) characterizes the growth rate of the HW entropy in the early (intermediate) time.

Time evolution of HW entropy per DOF



Comment on the gauge invariance

The HW entropy with the product ansatz is not gauge invariant.
Nevertheless we might expect that the gauge dependence is not serious.

Because

Growth rate of the HW entropy ≃ Sum of positive Lyapunov exponents

Gauge DOF dose not significantly contribute to chaoticity and instability.

[Iida et al. (2013)] [Tsutsui et al. (2015)]

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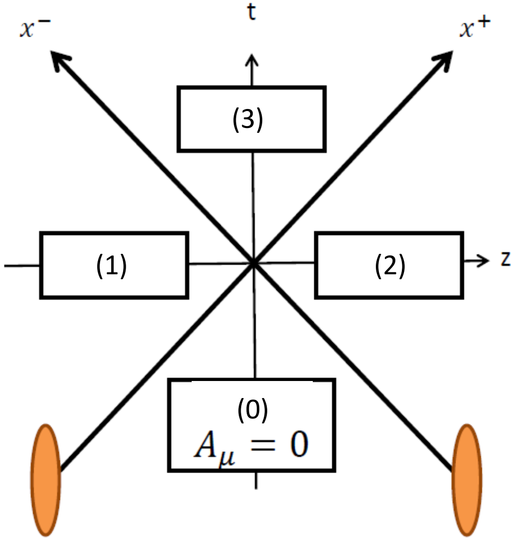
Phenomenological initial condition

H.T., T.Kunihiro, A.Ohnishi, and T.T.Takahashi, PTEP, 013D02 (2017).

Initial condition in heavy ion collision [Kovner-McLerran-Weigert(1995)]

$$[D_\nu, F^{\nu\mu}] = J^\mu,$$
$$J^\mu = \delta^{\mu +}\delta(x^-)\rho_1(x_\perp) + \delta^{\mu -}\delta(x^+)\rho_2(x_\perp),$$



In McLerran-Venugopalan(MV) model,

$$\langle \rho(\mathbf{x}_\perp)\rho(\mathbf{y}_\perp)\rangle = g^4\mu^2_{\rm phys}\delta^{(2)}(\mathbf{x}_\perp - \mathbf{y}_\perp),$$

[McLerran-Venugopalan(1993,1994,1994)]

MV configuration
(expanding geometry, τ-η coordinate)

$$\tau = \sqrt{t^2 - z^2}, \ \ \eta = \frac{1}{2}\ln\frac{t+z}{t-z}.$$

$$A^i_{\rm MV}, A^\eta_{\rm MV}, E^i_{\rm MV}, E^\eta_{\rm MV}$$

Physical scale a: lattice spacing, L: lattice size, R_A:radius of nucleus

$$\mu = Q_s = 2{\rm GeV}$$ (Gluon saturation scale) [Krasnitz-Nara-Venugopalan(2003)]

$$aL \simeq \sqrt{\pi}R_A \simeq 7\sqrt{\pi}\ [{\rm fm}] \qquad g = 1(\alpha_s = 0.08)$$

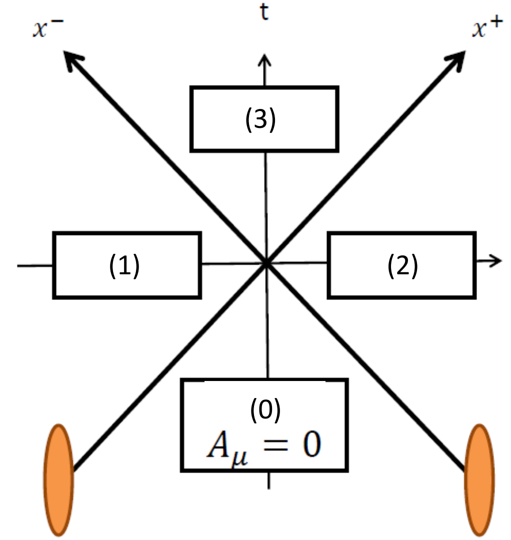
Phenomenological initial condition

H.T., T.Kunihiro, A.Ohnishi, and T.T.Takahashi, PTEP, 013D02 (2017).

Initial condition in heavy ion collision

[Kovner-McLerran-Weigert(1995)]

$$[D_\nu, F^{\nu\mu}] = J^\mu,$$
$$J^\mu = \delta^{\mu +}\delta(x^-)\rho_1(x_\perp) + \delta^{\mu -}\delta(x^+)\rho_2(x_\perp),$$



In McLerran-Venugopalan(MV) model,

$$\langle \rho(\mathbf{x}_\perp)\rho(\mathbf{y}_\perp)\rangle = g^4\mu^2_{\rm phys}\delta^{(2)}(\mathbf{x}_\perp - \mathbf{y}_\perp),$$

[McLerran-Venugopalan(1993,1994,1994)]

MV configuration

~~(expanding geometry, τ-η coordinate)~~(static geometry, xyz coordinate)

$$A^i_{\rm MV}, A^{z}_{\rm MV}, E^i_{\rm MV}, E^{z}_{\rm MV}$$

We mimic the MV configuration in the static box.
It is uniform in the z direction. [Iida et al(2014)]

Physical scale

a: lattice spacing, L: lattice size, R_A:radius of nucleus

$$\mu = Q_s = 2\mathrm{GeV}$$ (Gluon saturation scale) [Krasnitz-Nara-Venugopalan(2003)]

$$aL \simeq \sqrt{\pi}R_A \simeq 7\sqrt{\pi}\ [\mathrm{fm}] \qquad g = 1(\alpha_s = 0.08)$$

HW entropy production

H.T., T.Kunihiro, A.Ohnishi, and T.T.Takahashi, PTEP, 013D02 (2018).

Initial condition:

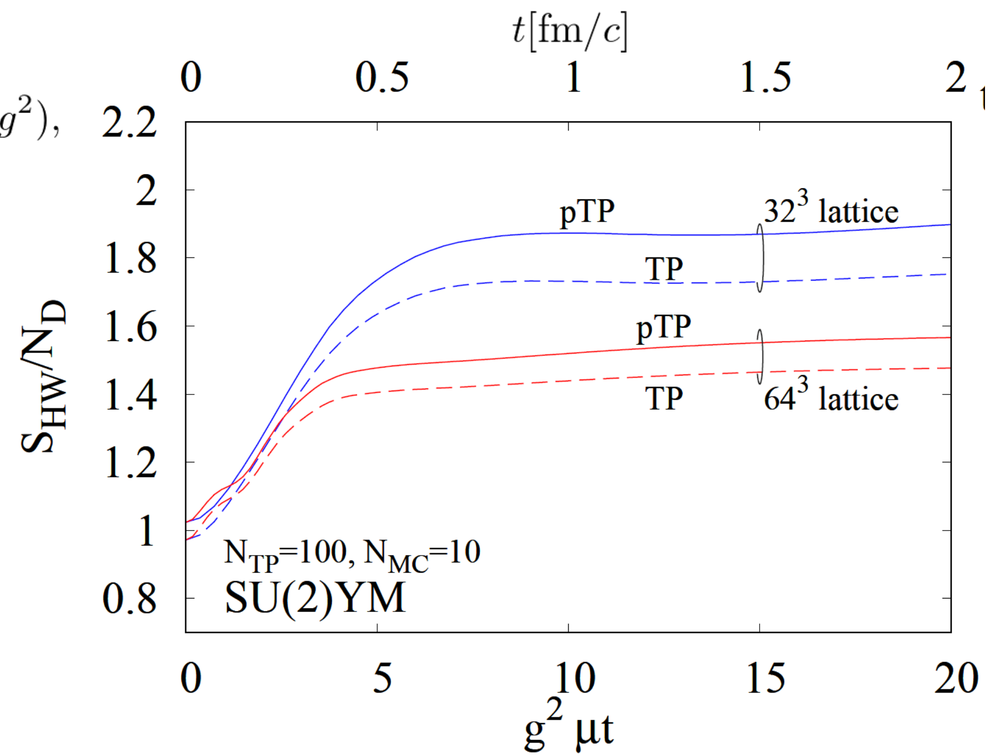
Gaussian around MV configuration

$$f_W[A, E; t=0] = 2^{N_D} \exp(-\omega(A - A_{\rm MV})^2/\hbar g^2 - (E - E_{\rm MV})^2/\omega\hbar g^2),$$

In the 64^3 lattice, the lattice spacing $a \simeq 2Q_s^{-1}$, which is about cutoff scale of semiclassical Yang-Mills field.

The entropy increases linearly in the early time.
The growth rates are same in the both 32^3 and 64^3 lattice.

Time evolution of HW entropy per DOF



The HW entropy is saturated around $g^2\mu t \sim 5 (t \sim 0.5 \text{fm}/c)$ on the 64^3 lattice.

HW entropy production

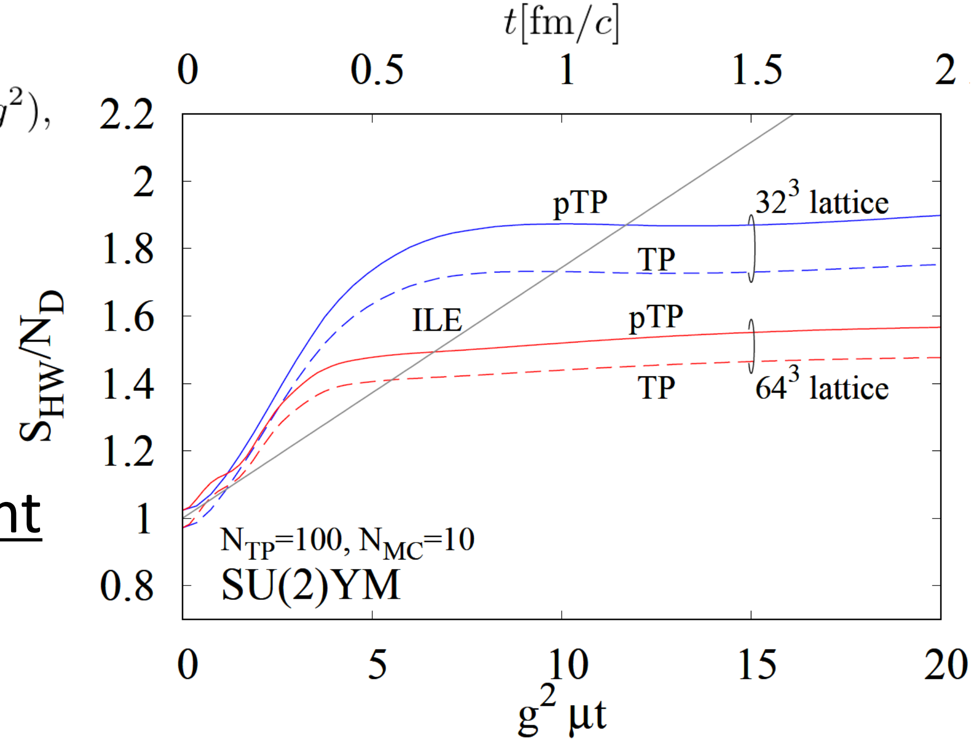
H.T., T.Kunihiro, A.Ohnishi, and T.T.Takahashi, PTEP, 013D02 (2018).

Initial condition:

Gaussian around MV configuration

$$f_{\rm W}[A,E;t=0]$$
$$= 2^{N_D}\exp(-\omega(A-A_{\rm MV})^2/\hbar g^2-(E-E_{\rm MV})^2/\omega\hbar g^2),$$

Time evolution of HW entropy per DOF



Compering with Lyapunov exponent

The growth rate of the HW entropy is larger than ILE.

We cannot explain the entropy production only by intrinsic chaoticity. It suggests that another possible machanism exists to create the HW entropy such as the initial instability.

Local energy distribution

H.T., T.Kunihiro, A.Ohnishi, and T.T.Takahashi, PTEP, 013D02 (2018).

Electric-(magnetic-) energy density

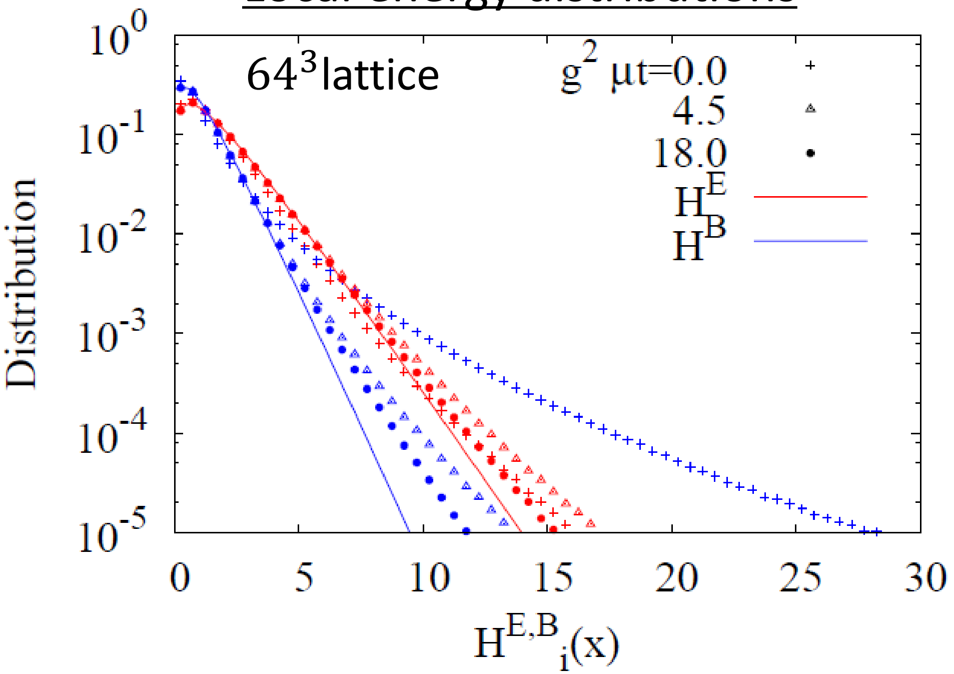
$$H_i^E(\mathbf{x}) = \frac{1}{2}\sum_{a=1}^{N_c^2-1}(E_i^a(\mathbf{x}))^2 \qquad H_i^B(\mathbf{x}) = \frac{1}{2}\sum_{a=1}^{N_c^2-1}(B_i^a(\mathbf{x}))^2$$

$$B^{ai} = -\frac{\epsilon^{ijk}}{2}F_{jk}^a$$

In the late time ($g^2\mu t = 4.5$),
the distributions reach the Boltzmann distributions(solid lines).

$$\sqrt{H^{E(B)}}\exp(-H^{E(B)}/T)$$

Local energy distributions



The magnetic and electric temperatures are different each other.
It might suggest the system reaches the quasi-equilibrium state.

Since the classical statistic distribution in field theory dose not have a well-defined continuum limit, it is reasonable to find the lattice size dependence of the saturation time and the saturation value of the HW entropy.

Pressure isotropization

H.T., T.Kunihiro, A.Ohnishi, and T.T.Takahashi, PTEP, 013D02 (2018).

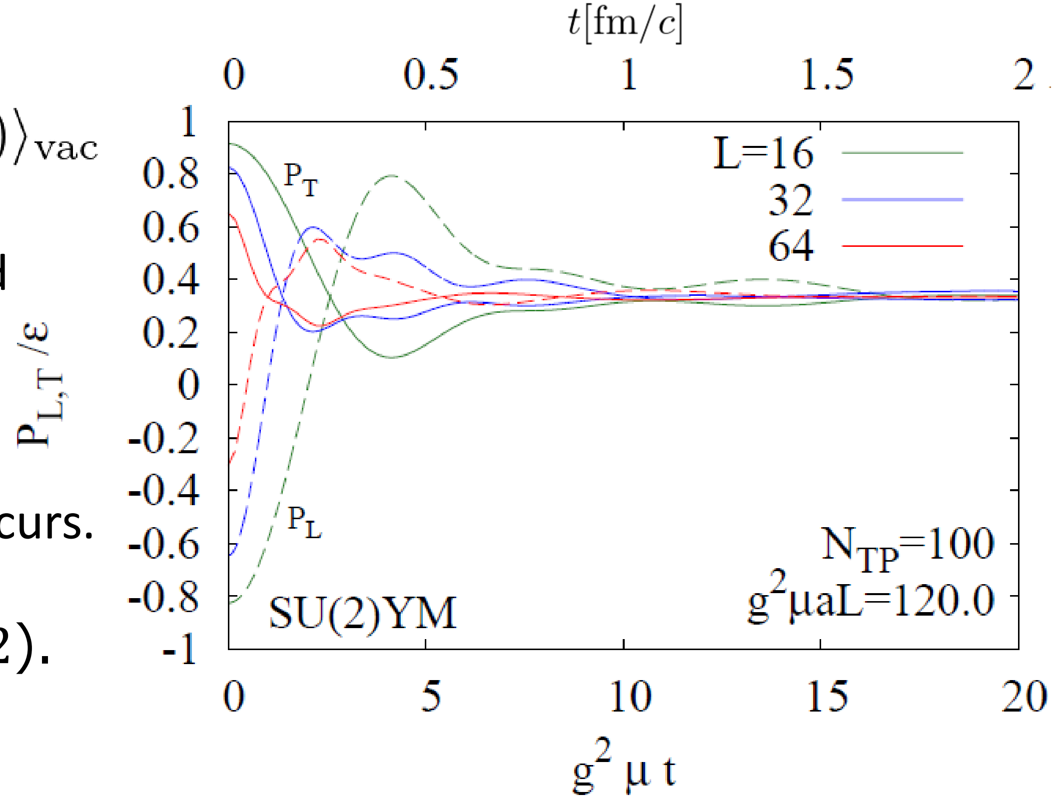
Vacuum subtraction

$$\langle X(t)\rangle = \langle X(t)\rangle_{\rm MV} - \langle X(t=0)\rangle_{\rm vac}$$

(Vacuum state is a Gaussian around the origin $\bar{A}, \bar{E} = 0$)

The isotropization of the pressure occurs.
The isotropization time converges $g^2\mu t \simeq 10$ for larger lattices ($L \geq 32$).

Time evolution of the pressure



The isotropization time roughly agrees with the time of the HW entropy saturation. It also happens to agree with the isotropization time obtained in the expanding geometry. [Epelbaum-Gelis(2013)]

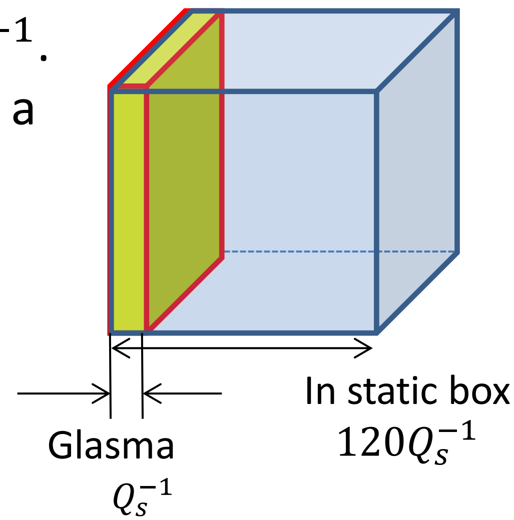
Discussion 1

H.T., T.Kunihiro, A.Ohnishi, and T.T.Takahashi, PTEP, 013D02 (2018).

The amount of the produced HW entropy $\frac{\Delta S}{N_D} \simeq 0.4$

Comparing with the phenomenological estimate

The longitudinal thickness of glasma at initial time is Q_s^{-1}. The present calculation in the static box corresponds to a very thick nuclei, $aL = 120Q_s^{-1}$.



The produced entropy per unit rapidity for color SU(3)

$$\frac{\Delta S_{\rm HW}}{120\Delta Y} = \frac{0.4 \times 3(N_c^2-1)L^3}{1200} \simeq 2000,$$

This value is the same order as the phenomenological entropy production. $\Delta S/\Delta Y \simeq 4500$ [Muller-Schafer(2011)]

Calculation of the entropy production in an expanding geometry is desired.

Discussion 2

Comparing with thermal entropy

Thermodynamic relation : $E - TS_{\mathrm{thermal}} + PV = 0,$

Energy: $E = \varepsilon V$

Temperature: $T = T_E$

Pressure per energy density: $\frac{P}{\varepsilon} \simeq 1/3$

$$\frac{S_{\mathrm{thermal}}}{N_D} \simeq 0.32,$$

It is the almost same as the amount of the increase of the HW entropy. $\frac{\Delta S}{N_D} \simeq 0.4$

It seems that the slightly overestimate is due to the product ansatz.

When the HW entropy saturates, the system nearly reaches the thermal equilibrium.

Discussion 3

Discussion on the saturation time of the HW entropy

The Boltzmann time $\frac{2\pi\hbar}{k_B T_E} = 0.54\ \mathrm{fm}/c$ is typical time scale of the system.

In the present calculation, it is accidentally the same as the saturation time of the HW entropy.

Relations with other contexts

- Isolated quantum systems

  The relaxation in Boltzmann time is shown in the discussion of the "typicality". It is pointed out that itself is a very early time scale. [Tasaki(2016)]

- Information paradox of a black hole

  The upper bound of the Lyapunov exponents, which characterizes the information loss, is predicted to be $2\pi k_B T$. [Maldacena-Shenker-Stanford(2016)]

  Ex.) Sachdev-Ye-Kitaev model Refs. are included in [Polchinski-Rosenhaus(2016)].

Summary

- To understand the thermalization in heavy ion collisions, we have analyzed Husimi-Wehrl(HW) entropy, which defined by quantum distribution function, in Yang-Mills field theory.

- We show the entropy production from the phenomenological initial condition, which mimics glasma in static box.
- We show that the saturation time of the HW entropy is around 1fm/c and it agrees with the equilibrium times of the local energy distributions and isotropization time of the pressure.

- The production rate of the HW entropy is significantly larger than the growth rate expected by the chaoticity.
- The amount of the produced entropy is around half of the expected entropy, $\frac{\Delta S}{\Delta Y} \simeq 4500$.