Status of viscous hydrodynamic code development

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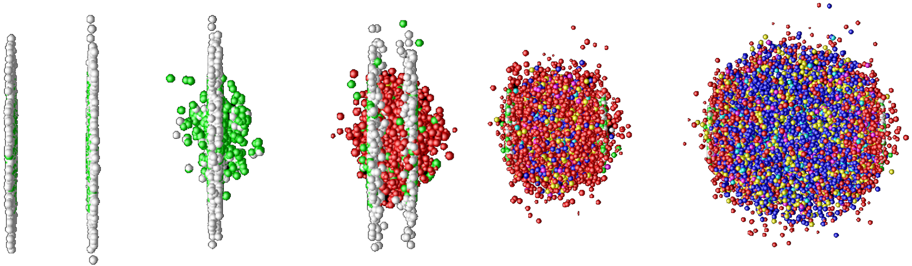
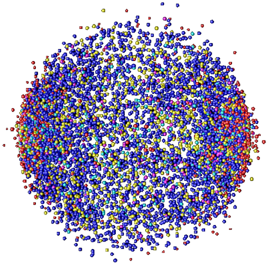
Introduction: heavy ion collision in pictures[1]



- Initial state, hard scatterings
- Thermalization
- Hydrodynamic expansion
  - Quark-gluon plasma
  - Phase transition
  - Hadron Gas
  - Chemical freeze-out
- Kinetic stage
- (kinetic) freeze-out

Typical size
10 fm $\propto$
10^{-14}m

Typical lifetime
10 fm/c $\propto$
10^{-23}s



[1] taken from event generator

Relativistic viscous hydrodynamics

$$\begin{cases} \partial_\mu T^{\mu\nu} = 0 \\ \partial_\mu N^\mu = 0 \end{cases}$$

$$T^{\mu\nu} = \varepsilon u^\mu u^\nu - (p+\Pi)\Delta^{\mu\nu} + q^\mu u^\nu + q^\nu u^\mu + \pi^{\mu\nu}$$

$$N^\mu = n \cdot u^\mu + v^\mu$$

+equation of state (EoS) $p = p(\varepsilon, n)$

where $\Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu$ is the projector orthogonal to u^μ,
(in the fluid rest frame where $u^\mu = (1,0,0,0) \Rightarrow \quad \Delta^{\mu\nu} = diag(0,-1,-1,-1)$)
q^μ is the heat flux
$\pi^{\mu\nu}$ and Π are the shear stress tensor and bulk pressure
v^μ is a (baryon, etc) charge diffusion

$q^\mu = \pi^{\mu\nu} = \Pi = v^\mu = 0 \Rightarrow$ ideal fluid.

Definition of u^μ

Since in general case there is no common direction for energy-momentum flow and particle flow,

1. Landau definition (flow of energy):

$$u_L^\mu = \frac{T^\mu_\nu u_L^\nu}{\sqrt{u_L^\alpha T_\alpha^\beta T_{\beta\gamma} u_L^\gamma}} \qquad \Rightarrow q^\mu = 0$$

2. Eckart definition (flow of conserved charge)

$$u_E^\mu = \frac{N^\mu}{\sqrt{N_\nu N^\nu}} \qquad \Rightarrow v^\mu = 0$$

! we adopt Landau definition (Eckart frame is undefined when $n = 0$)

Relativistic Navier-Stokes

from the second law of thermodynamics

In ideal hydrodynamics:

$$\left\{ \begin{array}{l} \partial_\mu T^{\mu\nu} = 0 \\ \partial_\mu N^\mu = 0 \end{array} \right. \qquad \Rightarrow \partial_\mu S^\mu = \partial_\mu (s u^\mu) = 0$$

In viscous hydrodynamics:

$$\partial_\mu S^\mu > 0$$

$$T \partial_\mu S^\mu = ... = -u_\nu \partial_\mu \delta T^{\mu\nu} = ... = \pi^{\mu\nu} \cdot < \Delta^{\mu\lambda} \partial_\lambda u^\nu > - \Pi \cdot \partial_\mu u^\mu$$

Then, $\partial_\mu S^\mu > 0$ if

$$\left\{ \begin{array}{l} \pi^{\mu\nu} = 2\eta < \Delta^{\mu\lambda} \partial_\lambda u^\nu > \\ \Pi = -\zeta \partial_\mu u^\mu \end{array} \right.$$

These constitutive equations correspond to relativistic generalization of Navier-Stokes equations.

Relativistic Navier-Stokes

Causality problems:
For small perturbations around uniform flow:

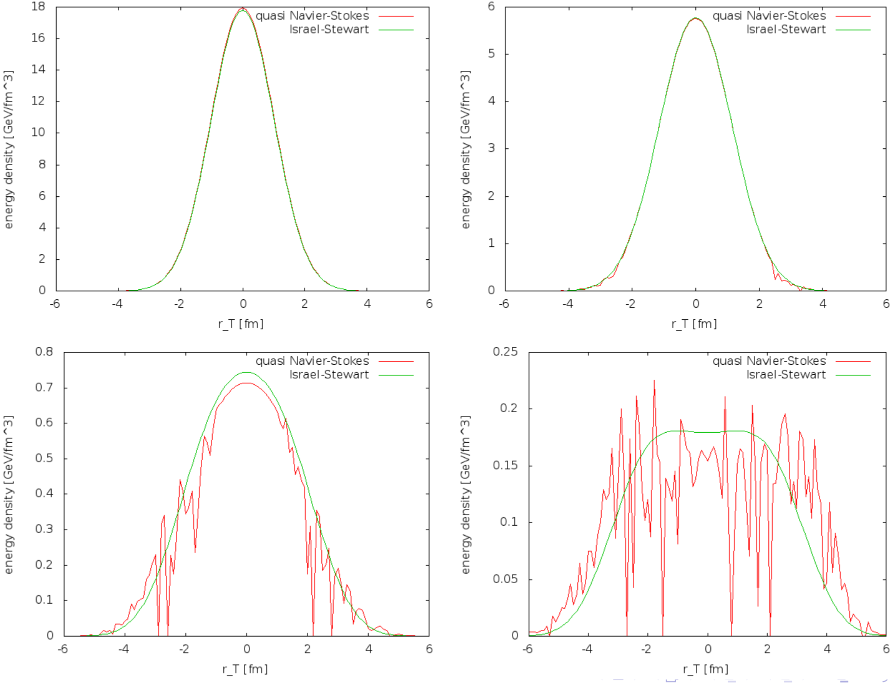
$$\partial_t \delta u^y - \frac{\eta}{\varepsilon + p} \partial_x^2 \delta u^y = 0$$

Diffusion speed for wavemode k:

$$v_T(k) = 2k \frac{\eta}{\varepsilon + p} \rightarrow \infty, \quad k \gg 1$$

Such paradox is a consequence of an insufficient description of the thermodynamical state in nonequilibrium (Müller, 1968)

Acausality $\rightarrow$ **instabilities** (Hiscock, Lindblom '1985)


quasi Navier-Stokes
Israel-Stewart
energy density [GeV/fm^3]
r_T [fm]
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quasi Navier-Stokes
Israel-Stewart
energy density [GeV/fm^3]
r_T [fm]

Second-order theory: Israel-Stewart formalism

Entropy current

$$S^\mu = su^\mu = s_{eq}u^\mu - (\beta_0\Pi^2 + \beta^2\pi^{\alpha\beta}\pi_{\alpha\beta})\frac{u^\mu}{2T}$$

Then, the requirement $\partial_\mu S^\mu > 0$ leads to

$$< D\pi^{\mu\nu} >= -\frac{\pi^{\mu\nu} - \pi^{\mu\nu}_{NS}}{\tau_\pi} - \frac{1}{2}\pi^{\mu\nu}\left(\partial_\lambda u^\lambda + D\ln\frac{\beta_0}{T}\right)$$

$$D\Pi = -\frac{\Pi - \Pi_{NS}}{\tau_\Pi} - \frac{1}{2}\Pi\left(\partial_\lambda u^\lambda + D\ln\frac{\beta_2}{T}\right)$$

where $D \equiv u^\mu\partial_\mu$, $\tau_\pi = 2\eta\beta_2$, $\tau_\Pi = \zeta\beta_0$, and Navier-Stokes values for the viscous components:

$$\pi^{\mu\nu}_{NS} = 2\eta\sigma^{\mu\nu} = \eta(\Delta^{\mu\lambda}\partial_\lambda u^\nu + \Delta\nu\lambda\partial_\lambda u^\mu) - \frac{2}{3}\eta\Delta^{\mu\nu}\partial_\lambda u^\lambda$$

$$\Pi_{NS} = -\zeta\partial_\lambda u^\lambda$$

The solutions are stable, provided that τ_π, τ_Π are big enough (for example, $\tau_\Pi > \frac{3}{2}\frac{\zeta}{sT}$ from stability condition around hydrostatic state*)

The scheme

Coordinate transformations

Milne coordinates

The coordinate system is defined as follows:

0) $\tau = \sqrt{t^2 - z^2}$

1) $x = x$

2) $y = y$

3) $\eta = \frac{1}{2} ln \frac{t+z}{t-z}$

$$g^{\mu\nu} = diag(1, -1, -1, -1/\tau^2)$$

Nonzero Christoffel symbols are:

$$\Gamma^{\eta}_{\tau\eta} = \Gamma^{\eta}_{\eta\tau} = 1/\tau, \quad \Gamma^{\tau}_{\eta\eta} = \tau$$

$$T^{\mu\nu} = (\varepsilon + p)u^\mu u^\nu - p \cdot g^{\mu\nu}, \text{ where}$$

$$u^\mu = \{\cosh(\eta_f - \eta)\cosh\eta_T, \sinh\eta_T\cos\phi, \sinh\eta_T\sin\phi, \frac{1}{\tau}\sinh(\eta_f - \eta)\cosh\eta_T\}$$

(cf. $u^i_{\text{Cart}} = \{\cosh(\eta_f)\cosh\eta_T, \sinh\eta_T\cos\phi, \sinh\eta_T\sin\phi, \sinh(\eta_f)\cosh\eta_T\}$)

EM conservation equations are

$$\partial_{;\nu} T^{\mu\nu} = 0$$

or

$$\mu = 0: \quad \partial_\nu T^{\tau\nu} + \tau T^{\eta\eta} + \frac{1}{\tau} T^{\tau\tau} = 0$$

$$\mu = 1: \quad \partial_\nu T^{x\nu} + \frac{1}{\tau} T^{x\tau} = 0$$

$$\mu = 2: \quad \partial_\nu T^{y\nu} + \frac{1}{\tau} T^{y\tau} = 0$$

$$\mu = 3: \quad \partial_\nu T^{\eta\nu} + \frac{3}{\tau} T^{\eta\tau} = 0$$

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$$\mu = 2: \quad \partial_\nu T^{y\nu} + \frac{1}{\tau} T^{y\tau} = 0$$

$$\mu = 3: \quad \partial_\nu T^{\eta\nu} + \frac{3}{\tau} T^{\eta\tau} = 0$$

Additional transformations:

$T^{\mu\eta} \to T^{\mu\eta}/\tau,\ \mu \neq \eta,$
$T^{\eta\eta} \to T^{\eta\eta}/\tau^2$

$$\Downarrow\Downarrow$$

$$\partial_\nu(\tau T^{\tau\nu}) + \frac{1}{\tau}(\tau T^{\eta\eta}) = 0$$

$$\partial_\nu(\tau T^{x\nu}) = 0$$

$$\partial_\nu(\tau T^{y\nu}) = 0$$

$$\partial_\nu(\tau T^{\eta\nu}) + \frac{1}{\tau}\tau T^{\eta\tau} = 0$$

Conservative variables are
$Q^\mu = \tau \cdot T^{\tau\mu}$

The exact expressions for **evolutionary equations** for viscous corrections:

$$\gamma(\partial_t + v_i \partial_i)\,\pi^{\mu\nu} = -\frac{\pi^{\mu\nu} - \pi^{\mu\nu}_{\text{NS}}}{\tau_\pi} + I_\pi \quad (1)$$

$$\gamma(\partial_t + v_i \partial_i)\,\Pi = -\frac{\Pi - \Pi_{\text{NS}}}{\tau_\Pi} + I_\Pi \quad (2)$$

where the extra source-terms are:

$$I_\pi = -\overbrace{\frac{4}{3}\pi^{\mu\nu}\partial_{;\gamma}u^\gamma}^{\text{extra IS terms}} - \overbrace{[u^\nu \pi^{\mu\beta} + u^\mu \pi^{\nu\beta}]u^\lambda \partial_{;\lambda} u_\beta}^{<D\pi^{\mu\nu}>=...} + I_{\pi,\text{geom}}(\pi) \quad (3)$$

$$I_\Pi = -\frac{4}{3}\Pi\partial_{;\gamma}u^\gamma + I_{\Pi,\text{geom}}(\Pi) \quad (4)$$

and $\partial_{;\mu} u^\nu = \partial_\mu u^\nu + \Gamma^\nu_{\mu\lambda} u^\lambda$ is covariant derivative.

Closer to numerics:

$$\partial_\mu (T^{\mu\nu}_{\rm id} + \delta T^{\mu\nu}) = S^\nu, \qquad \text{S=geometrical source terms}$$

$$\partial_\tau \underbrace{(T^{\tau i}_{\rm id} + \delta T^{\tau i})}_{Q_i} + \partial_j \underbrace{(T^{ji})}_{\text{id.flux}} + \partial_j \underbrace{(\delta T^{ji})}_{\text{visc.flux}} = \underbrace{S^\nu_{\rm id} + \delta S^\nu}_{\text{source terms}}$$

Finite-volume realization:

$$\frac{1}{\Delta\tau}(Q^{n+1}_{\rm id} + \delta Q^{n+1} - Q^n_{\rm id} - \delta Q^n) + \frac{1}{\Delta x}(\Delta F^{n+1/2}_{\rm id} + \Delta\delta F^{n+1/2}) = S^{n+1/2}_{\rm id} + \delta S^{n+1/2}$$

[2] Makoto Takamoto, Shu-ichiro Inutsuka, J.Comput.Phys. 230 (2011), 7002

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now, a small trick:

$$\frac{1}{\Delta\tau}(Q^{n+1}_{\rm id} + \delta Q^{n+1} \underbrace{-Q^{*n+1}_{\rm id} + Q^{*n+1}_{\rm id}}_{=0} - Q^n_{\rm id} - \delta Q^n) + \frac{1}{\Delta x}(\Delta F_{\rm id} + \Delta\delta F) = S_{\rm id} + \delta S$$

Then,split the equation into two parts[2]:

$$\frac{1}{\Delta t}(Q^{*n+1}_{\rm id} - Q^n_{\rm id}) + \frac{1}{\Delta x}\Delta F_{\rm id} = S_{\rm id} \tag{5}$$

$$\frac{1}{\Delta t}(Q^{n+1}_{\rm id} + \delta Q^{n+1} - Q^{*n+1}_{\rm id} - \delta Q^n) + \frac{1}{\Delta x}\Delta\delta F = \delta S \tag{6}$$

[2] Makoto Takamoto, Shu-ichiro Inutsuka, J.Comput.Phys. 230 (2011), 7002

The solution then proceeds in two stages:

1a) Q_{id}^{*n+1} is obtained by evolving only the ideal part (5) of energy-momentum tensor over the full timestep Δt, which is done accurately using Godunov method: rHLLE flux + MUSCL + predictor-corrector schemes

$$\Rightarrow \quad \partial_\tau u^\nu \simeq \frac{u^{*(n+1),\nu} - u^{n,\nu}}{\Delta\tau} \qquad \partial_x u^\nu \simeq \frac{u_{k+1}^{*(n+1),\nu} - u_{k-1}^{*(n+1),\nu}}{2\Delta x}$$

1b) Advection-type evolutionary equations for $\pi^{\mu\nu}, \Pi$ are solved with 1st order upwind scheme. Velocity gradients for Navier-Stokes values of $\pi^{\mu\nu}, \Pi$ etc. are taken from above.

2) $Q_{\text{id},i}^{n+1} + \delta Q_i^{n+1} = Q_{\text{full}}^{n+1}$ is obtained from Eq. (6) evolving over the full timestep Δt with viscous fluxes/sources ONLY.

*The initial condition for stage 2 is $Q_{\text{ini}} = Q_{\text{id}}^{*n+1} + \delta Q^n$, the first term obtained from the solution of the stage 1.

test #0: 0+1D

comparison with known analytical solution with viscosity in Navier-Stokes limit

Energy conservation: $\partial_\nu T^{\tau\nu} + \tau T^{\eta\eta} + \frac{1}{\tau} T^{\tau\tau} = 0$

0+1D, $u^\mu = 1,0,0,0$, $T^{\mu\nu}_{\mathrm{id}} = diag(\varepsilon, p, p, p/\tau^2)$,
the only nonzero $\pi^{\mu\nu}$ is $\tau^2 \pi^{\eta\eta} = -\frac{4}{3}\frac{\eta}{\tau}$

$$\frac{\partial \varepsilon}{\partial \tau} + \frac{\varepsilon + p + \tau^2 \pi^{\eta\eta}}{\tau} = 0$$

Assuming the EoS for relativistic maseless gas, $\varepsilon = \alpha T^4$, $s = \frac{4}{3}\frac{\varepsilon}{T}$,
the solution is:

$$T(\tau) = \left(\frac{\tau_0}{\tau}\right)^{1/3} \left[T(\tau_0) + \frac{2\eta}{3s\tau_0} \left(1 - \left(\frac{\tau_0}{\tau}\right)^{2/3} \right) \right]$$

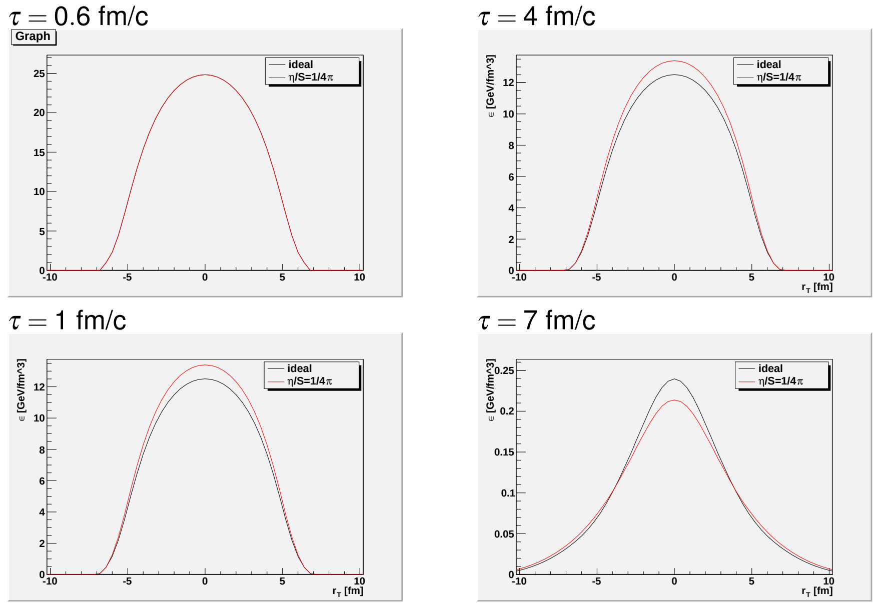
The same solution exists for bulk viscosity

setup for test calculations[3] #1

- Glauber IC for energy density
- $\tau_0 = 0.6$ fm/c, longitudinal boost-invariance
- $p = \varepsilon/3$ EoS
- Navier-Stokes values for initial $\pi^{\mu\nu}$ (nonzero due to Bjorken longitudinal flow)
- $\eta/s = 1/(4\pi)$, $\zeta/s = 0$ compared to ideal case

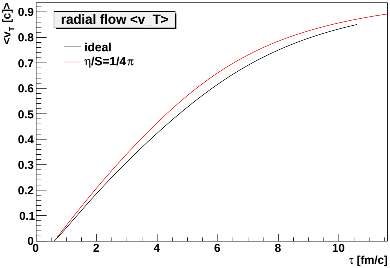
[3] to compare with H. Song, PhD thesis, arXiv:0908.3656

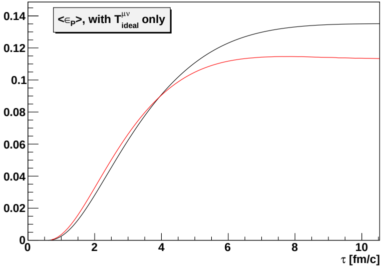
test #1: cooling rates for ideal/visc

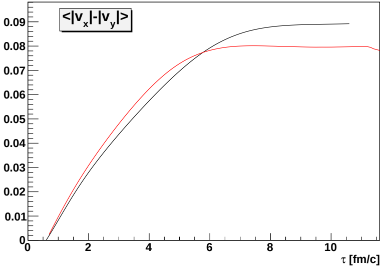


test #1: additional transverse push & anizotropy suppression

Central and noncentral collisions:
Glauber IC, $b = 7$ fm







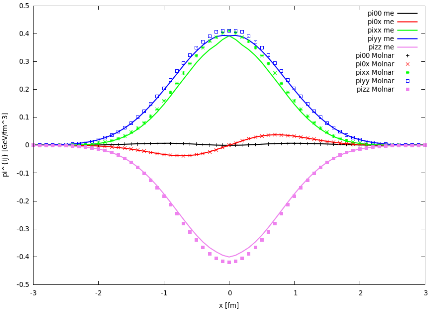
test #2

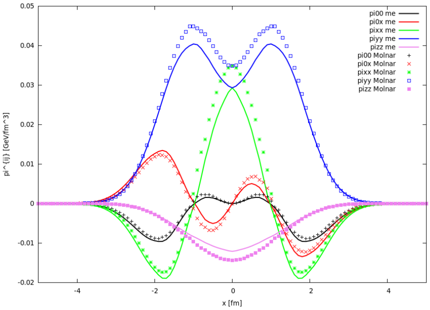
Comparison with vSHASTA by E. Molnar

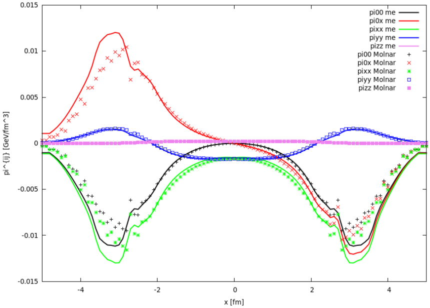
- 3D Gaussian for ε, $\tau_0 = 1$ fm/c
- EoS $p = \varepsilon/3$
- $\pi^{\mu\nu}_{\rm ini} = 0, \Pi_{\rm ini} = 0$
- no vacuum cells

Energy density/velocity proifles agree

$\pi^{\mu\nu}$ **plotted at:** 1.4 (left), 2.2 (top right), 3.8 fm/c (bottom right)



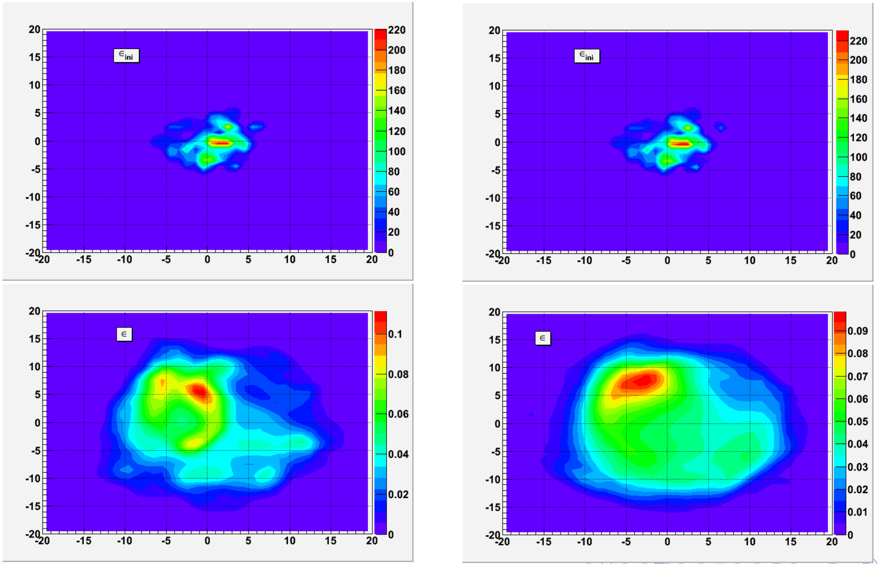




Test #3: Fluctuating ICs: (typical EPOS event)

Ideal hydro

$\eta/S = 3/(4\pi)$



- Energy conservation checks has been made as well (trivial for $\eta/s = 0$ in Cartesian coordinates, nontrivial otherwise)
- Some speed and memory usage optimization, cleanup of the code, improving the structure etc. needs to be done

Stay tuned!

Thanks for your attention!