A Closer Look on the Gunion-Bertsch Approximation

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The Context - Partonic Transport Model BAMPS

BAMPS = Boltzmann Approach to Multiple Particle Scattering [1]

Microscopic transport simulations with full dynamics

Attack various problems within *one* model.
(elliptic flow, R_{AA}, thermalization, ...)

Solve Boltzmann equation for $2 \rightarrow 2$ *and* $2 \leftrightarrow 3$ processes based on LO pQCD matrix elements.

$$p^{\mu}\partial_{\mu}f\,(x,p) = \mathcal{C}_{2\rightarrow 2}\,(x,p) + \mathcal{C}_{2\leftrightarrow 3}\,(x,p)$$

[1] Z. Xu, C. Greiner, Phys. Rev. C71 (2005)

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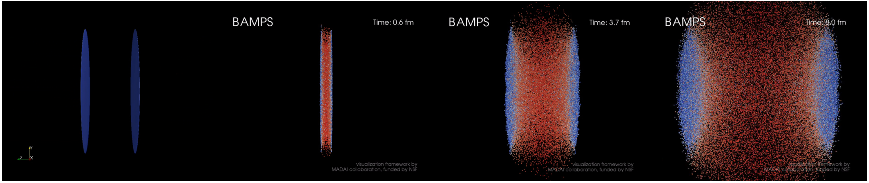

Visualization by Jan Uphoff
Visualization framework courtesy MADAI collaboration
funded by the NSF under grant NSF-PHY-09-41373

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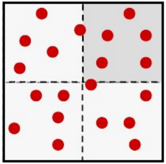
Monte Carlo sampling of interactions

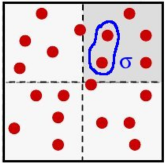
- Boltzmann particles
  - Massless for gluons and light quarks
  - Massive for heavy quarks
- Discretize:
  - Spatial cells ΔV
  - Time steps Δt
- Use testparticle method for sufficient statistics

$$N \rightarrow N \cdot N_{\text{test}}$$

- Sampling of interaction probabilities from x-sections

$$P_{2N} = v_{\text{rel}} \sigma_{2N} \frac{1}{N_{\text{test}}} \frac{\Delta t}{\Delta V} \qquad P_{32} = \frac{1}{8 E_A E_B E_C} I_{32} \frac{1}{N_{\text{test}}^2} \frac{\Delta t}{(\Delta V)^2}$$





The Context - Partonic Transport Model BAMPS

Monte Carlo sampling of interactions

- Sampling of interaction probabilities from LO pQCD
  - $2 \rightarrow 2$ Small angle cross sections
  - $2 \leftrightarrow 3$ Gunion-Bertsch matrix element
- Cross sections screened with dynamically computed Debye mass
  $m_D^2 = d_G \pi \alpha_s \int \frac{d^3p}{(2\pi)^3} \frac{1}{p} \left(N_c f_g + N_f f_q \right)$
- α_s either fixed (most of this talk) or running (heavy quarks)

$gg \rightarrow gg$ cross section

$$\frac{d\sigma_{gg\rightarrow gg}}{dq_\perp^2} \simeq \frac{9\pi\alpha_s^2}{2(\mathbf{q}_\perp^2 + m_D^2)^2}$$

Gunion-Bertsch matrix element

$$\left|\mathcal{M}_{gg\rightarrow ggg}\right|^2 = \frac{72\pi^2\alpha_s^2 s^2}{(\mathbf{q}_\perp^2 + m_D^2)^2} \frac{48\pi\alpha_s \mathbf{q}_\perp^2}{\mathbf{k}_\perp^2 \left[(\mathbf{k}_\perp - \mathbf{q}_\perp)^2 + m_D^2\right]}$$

Approximation vs. Exact Radiation Amplitude

Gunion and Bertsch approximated the LO radiation amplitude

Phys.Rev.,D25 (1982)

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The exact result is also known

Berends et al., PLB 103 (1981); Ellis and Sexton, Nucl.Phys.,B269 (1986)

$$\begin{aligned}|M_{\text{exact}}|^2 = \frac{g^6}{2}\left[N^3/(N^2-1)\right] &[(12345)+(12354)+(12435)+(12453)+(12534)\\
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- GB has been widely used for e.g. rate equations due to its simplicity
- How good is this approximation?

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A Recently Revived Debate

- J.-W. Chen, J. Deng, H. Dong, Q. Wang claim:

  BAMPS results are off by a factor 6 *due to miscounting of symmetry factors* arXiv:1107:0522

- B. Zhang analyzes GB vs. exact and finds differences up to 50% arXiv:1208.1224

GB - good, ok, really bad? Did we miscount symmetry factors?

- Extensive numerical comparisons between Gunion-Bertsch and exact matrix elements
- Analytically re-visit the derivation of the Gunion-Bertsch result

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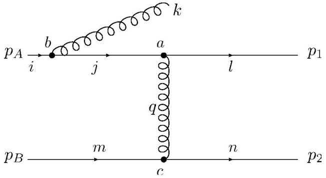
A Recently Revived Debate

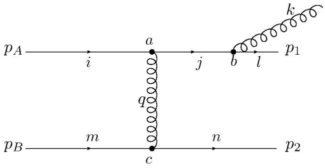
The short version

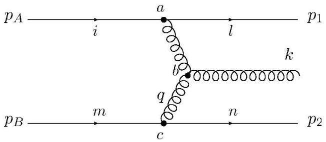
- Yes, there is a discrepancy between Gunion-Bertsch and the exact matrix element in some regions of the phase space
- It is not caused by symmetry factors but lies deeper within the approximations
  - The findings of Chen et al. are coincidental
  - Their reasoning does not hold
  - In BAMPS the discrepancy is probably at most a factor 3 as restrictions on the elastic part are already included
- Screening has an influence on the quality of the approximation (cf. Chen et al vs. Zhang), more later

Beware: Work in progress!

Diagrams:







plus radiation from lower lines . . .

Kinematics: (light-cone coordinates)

$p_A = (\sqrt{s}, 0, 0, 0) \qquad p_B = (0, \sqrt{s}, 0, 0)$

$k = (x\sqrt{s}, \frac{k_\perp^2}{x\sqrt{s}}, \mathbf{k}_\perp) \qquad q = (q^+, q^-, \mathbf{q}_\perp)$

Momentum conservation gives

$p_1 = p_A + q - k \qquad p_2 = p_B - q$

- k = momentum of radiated gluon, q = exchanged momentum
- Gunion-Bertsch: $A^+ = 0$ gauge, lower lines do not contribute (much)
- Scalar QCD to simplify calculations

Gunion-Bertsch Basics

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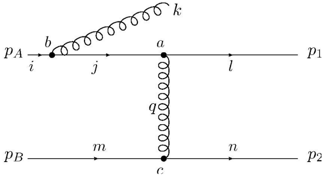

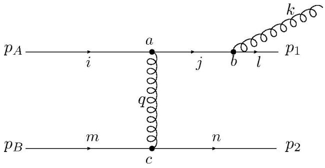

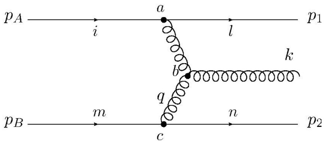

Rapidity of emitted gluon

$$y = \frac{1}{2} \ln \frac{k^+}{k^-} = \ln \frac{x\sqrt{s}}{k_\perp}$$

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The Problems with Gunion-Bertsch

Gunion and Bertsch explicitly state the following approximations:
$k_\perp \ll \sqrt{s}$, $q_\perp \ll \sqrt{s}$, $xq_\perp \ll k_\perp$

So where are the problems?

- A missing $(1-x)^2$ term

$$|\mathcal{M}_{GB}|^2 \sim (1-x)^2 \frac{s^2}{\mathbf{q}_\perp^2} \frac{1}{\mathbf{k}_\perp^2 (\mathbf{k}_\perp - \mathbf{q}_\perp)^2}$$

  x is the fraction of forward-momentum carried by the radiated gluon, $x = \frac{k_\perp}{\sqrt{s}} e^y$

- When not at midrapidity, $y = 0 \Rightarrow x = \frac{k_\perp}{\sqrt{s}}$, constraints are needed to arrive at the GB result that break the symmetry and make it only valid for forward emission

$$k_\perp^2 \ll x^2 s \quad \Rightarrow \quad k^+ \gg k^- \quad \Rightarrow \quad y \gg 0$$

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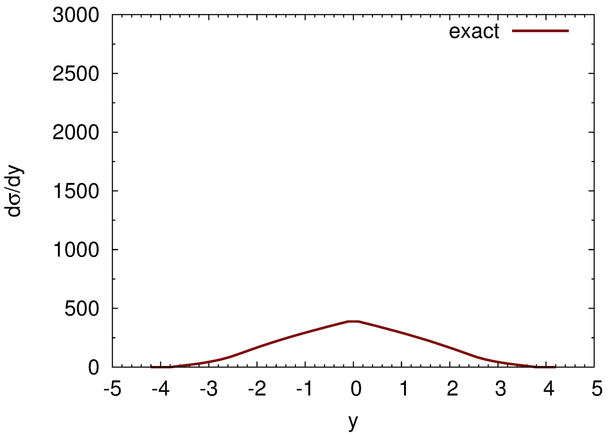
The Differential $qq \to qqg$ Cross Section



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- Integration both in GB coordinates and in standard phase space with numeric δ-functions

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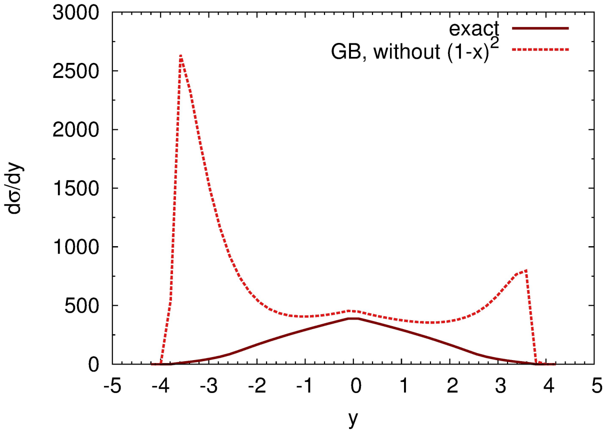

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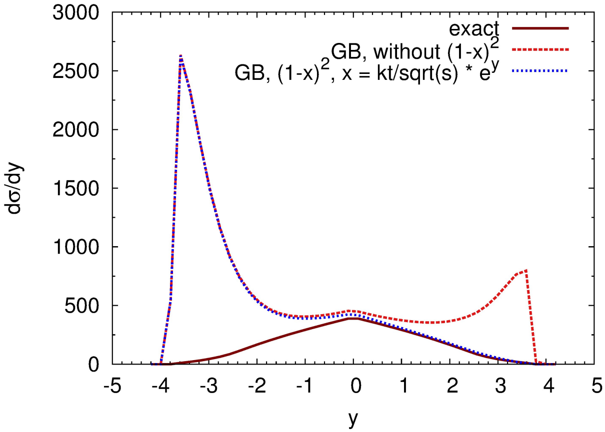

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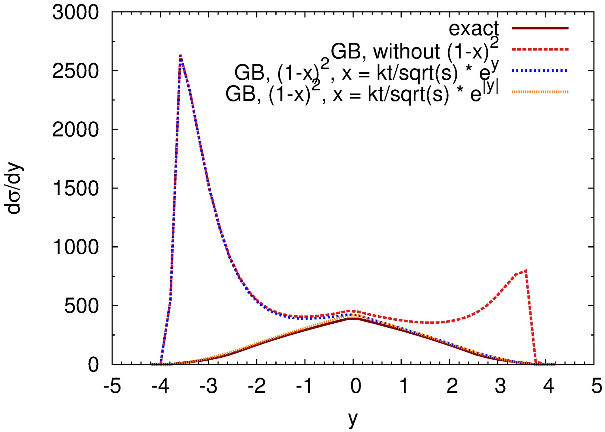

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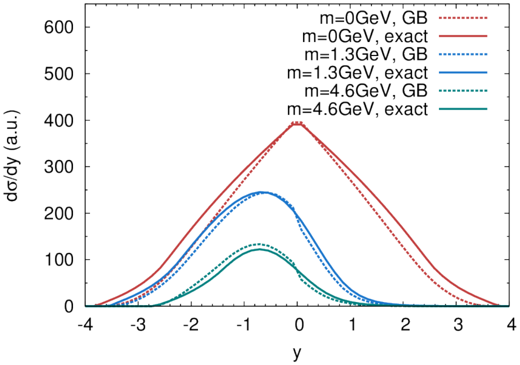
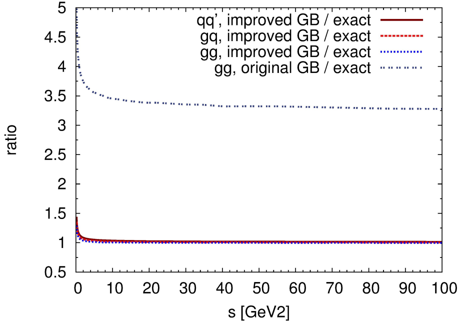
The Differential Heavy Quark Cross Section

Extending Gunion-Bertsch to finite masses including the corrections and comparing to the known exact results Kunszt, Pietarinen, Reya, PRD (1980)



- Gunion-Bertsch approximations including the corrections also work for heavy quarks!
- Asymmetry due to dead cone effect nicely visible

Comparing the Approximation to the Exact Results



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- GB only looked at the emission spectra at midrapidity, there the approximations are ok

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- Corrections for the total cross section *and* the kinematic sampling

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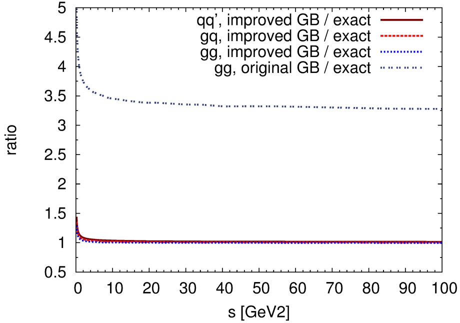

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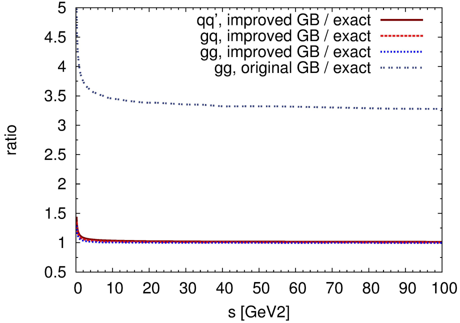

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Impact of Screening

Remember: Exact ME for $gg \to ggg$

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- With $\lambda = \epsilon m_D^2$
- So far: $\epsilon \ll 1$
- Systematic comparison but artificial screening (non-physical cross sections)

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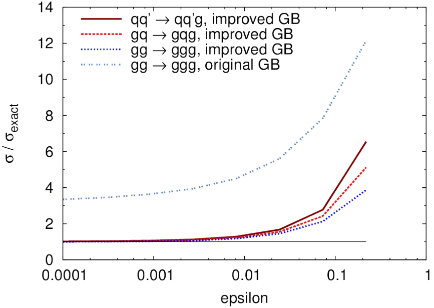
Quality of GB When Evolving the Infrared-Cutoff



The larger the cutoff, the worse the approximation. Large λ cut away the parts where GB is good...

Estimate the physical cutoff

1. Compute $d\sigma/dy$ at $y = 0$ with improved GB and standard Debye screening
2. Vary ϵ to get the same $d\sigma/dy$ for improved GB with cutoff scheme

Yields $\epsilon_{phys} \approx 0.3 \Rightarrow \sigma_{GB}/\sigma_{exact} \approx 2-4$

Can this be cured? Not quite sure yet.

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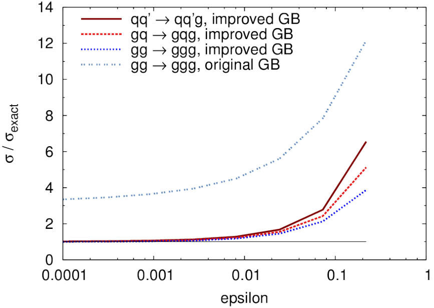

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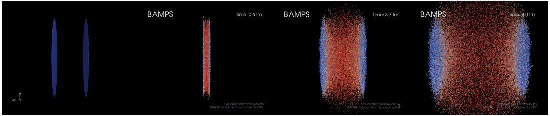
Summary

- Gunion-Bertsch needs to be improved when evaluating cross sections
- Improvements affect total cross section and momentum sampling
- In principle the improved GB approximates the exact results extremely well
- Physical screening might reduce the agreement



Implementation into BAMPS and investigation of effects on observables is underway. First results:

- Qualitatively good for high-p_T, cures peculiar energy loss features
- Implications stronger for high-p_T than for medium particles