Canonical suppression & enhancement using Master equations

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Outline

1. Motivation
2. Master equations
3. Canonical suppression in $\pi\pi \leftrightarrow K\overline{K}$ transition
4. Canonical enhancement in $c\bar{c} \leftrightarrow J/\Psi$ transition
5. Outlook!

Motivation

- **Particle number** important observable
- **Statistical calculations** relevant for comparing with experiment and microscopical simulations
  - Differences between **canonical** and **grand canonical** approach?
- Usage of **Master equation**:
  - Numerically rather simple
  - Adaptable to different transitions

Master equation

Differential equation providing the **time development of the probabilities** in a system.

General structure of a Master equation

$$\frac{\mathrm{d}P_i}{\mathrm{d}t} = \sum_j \left(\alpha_{j\to i} P_j - \alpha_{i\to j} P_i\right) \tag{1}$$

P_j probability of the system being in state j.
$a_{j\to i}$ transition rate from state j to state i.

Simulation setup (canonical suppression)

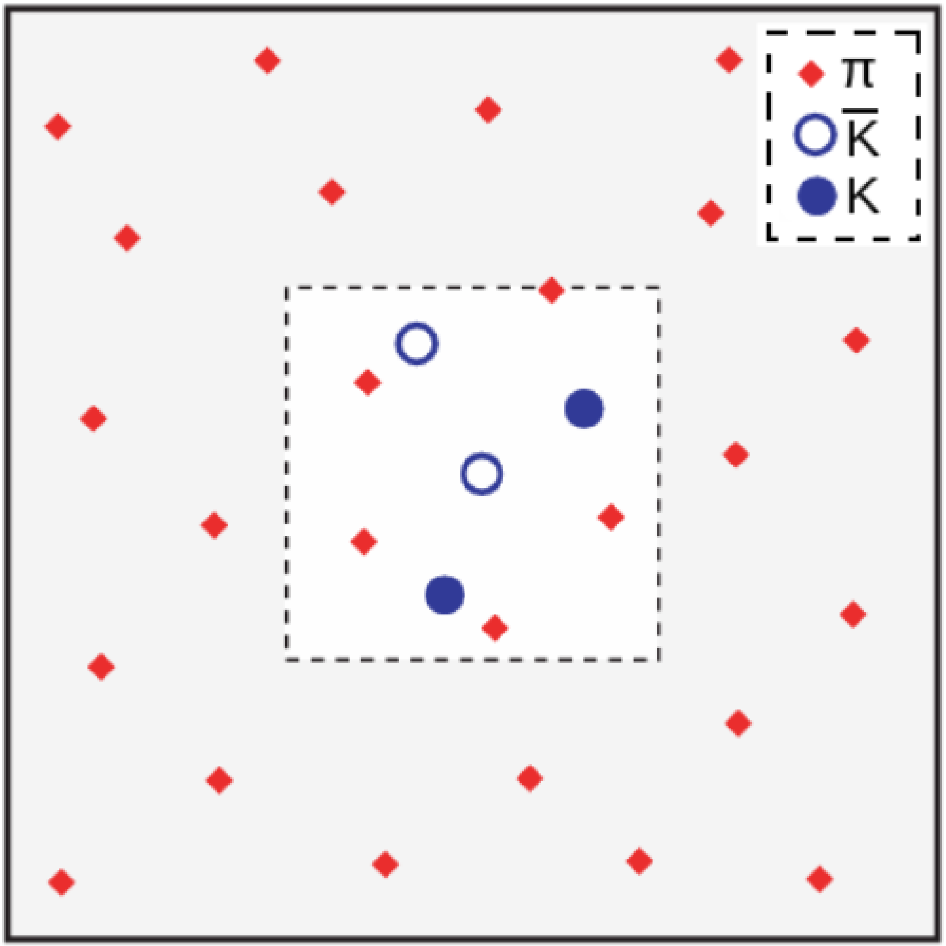

Figure: Simulation setup for Kaon production [Phys. Rev. C, 74:034902]

Canonical suppression

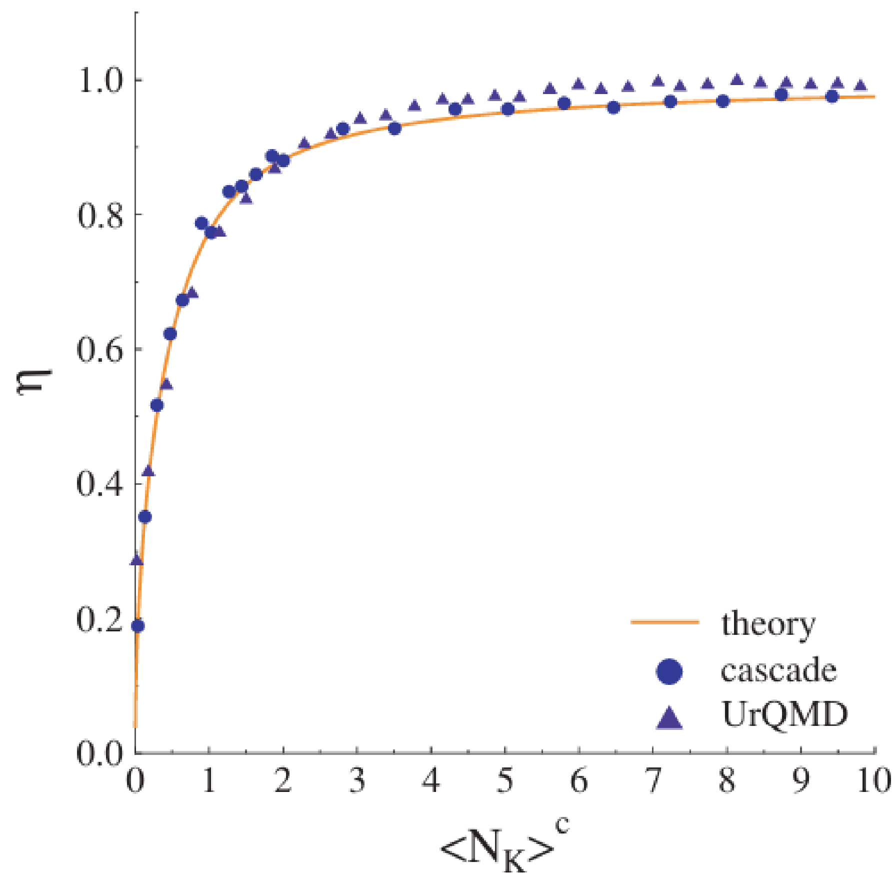

Figure: Canonical suppression [Phys. Rev. C, 74:034902]

- Canonical approach demands **exact conservation**
  - Reduces the phase space for transition
- Suppressed production of particles in canonical approach compared to grand-canonical

Suppression factor

$$N_{eq}^{C} = N_{eq}^{GC} \frac{I_1(2N_{eq}^{GC})}{I_0(2N_{eq}^{GC})} = N_{eq}^{GC}\eta$$

$$\eta \equiv \frac{N_{eq}^{C}}{N_{eq}^{GC}} = \frac{I_1(2N_{eq}^{GC})}{I_0(2N_{eq}^{GC})} \qquad (2)$$

Master equation for $\pi\pi \leftrightarrow K\overline{K}$ transition

$P_i \to P_n$ probability for a system with n Kaons and Antikaons
Transitions are of type $\alpha_{n\to n+1}$, $\alpha_{n\to n-1}$, a particle is **produced** or **decays**

Adapted Master equation

$$\frac{\mathrm{d}P_n}{\mathrm{d}t} = \alpha_{n-1\to n}P_{n-1} + \alpha_{n+1\to n}P_{n+1} - (\alpha_{n\to n+1} + \alpha_{n\to n-1})P_n \tag{3}$$

Transition rates

α can be divided into two parts:

Transition rate of one pair

$\mathit{\Gamma}_G$ rate for $\pi\pi \rightarrow K\overline{K}$ formation $(\alpha_{n\rightarrow n+1})$
$\mathit{\Gamma}_L$ rate for $K\overline{K} \rightarrow \pi\pi$ decay $(\alpha_{n\rightarrow n-1})$

$$\mathit{\Gamma} \equiv \frac{\mathit{\Gamma}_G}{\mathit{\Gamma}_L} = \frac{Z^1_K Z^1_{\overline{K}}}{Z^1_\pi Z^1_\pi} \tag{4}$$

Number of possible transitions

N^2_{pair} possible combinations.

Master equation for $\pi\pi \leftrightarrow K\overline{K}$

Time development of Kaon number probability

$$\begin{aligned}\frac{\mathrm{d}P_n}{\mathrm{d}t} &= \Gamma_G \langle N_\pi \rangle \langle N_\pi \rangle [P_{n-1} - P_n] - \Gamma_L [n^2 P_n - (n+1)^2 P_{n+1}] \\ \frac{\mathrm{d}P_n}{\mathrm{d}\tau} &= \Gamma \langle N_\pi \rangle \langle N_\pi \rangle [P_{n-1} - P_n] - [n^2 P_n - (n+1)^2 P_{n+1}]; \quad \tau = \Gamma_L t\end{aligned} \tag{5}$$

Kaon number in equilibrium

$$\langle N_K^{eq} \rangle = \langle N_{\overline{K}}^{eq} \rangle = \sum_n n P_n$$

Comparing the results

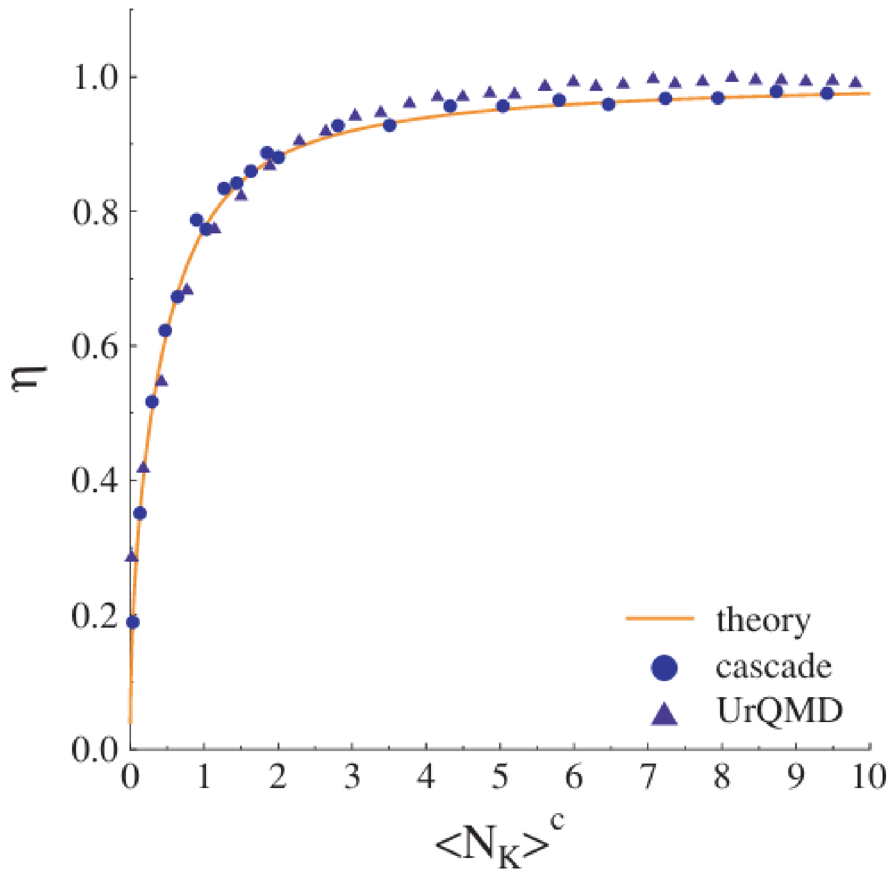

Figure: Microscopical simulation

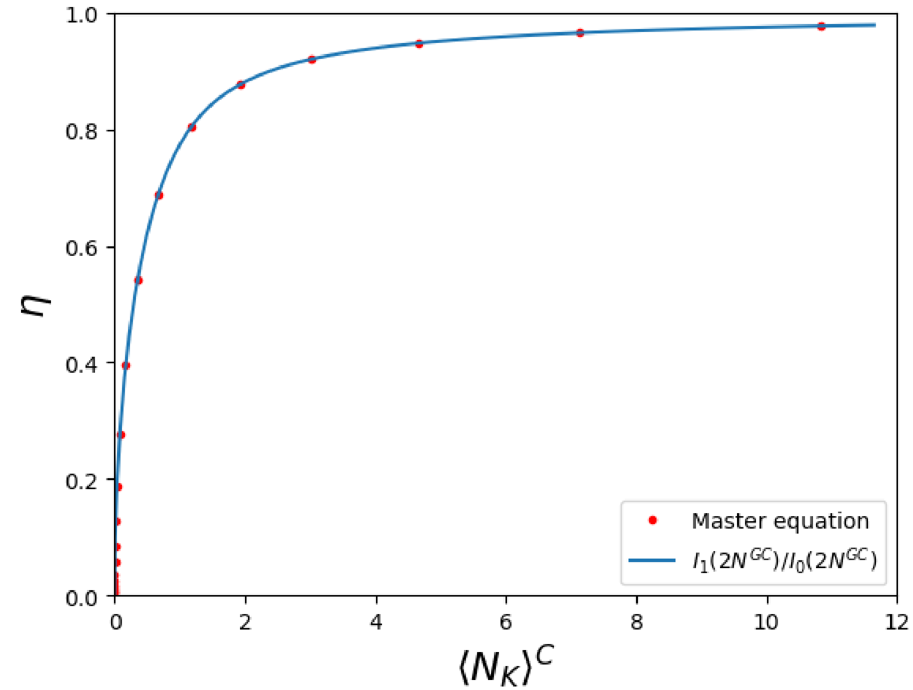

Figure: Master equation

Naomi's simulation

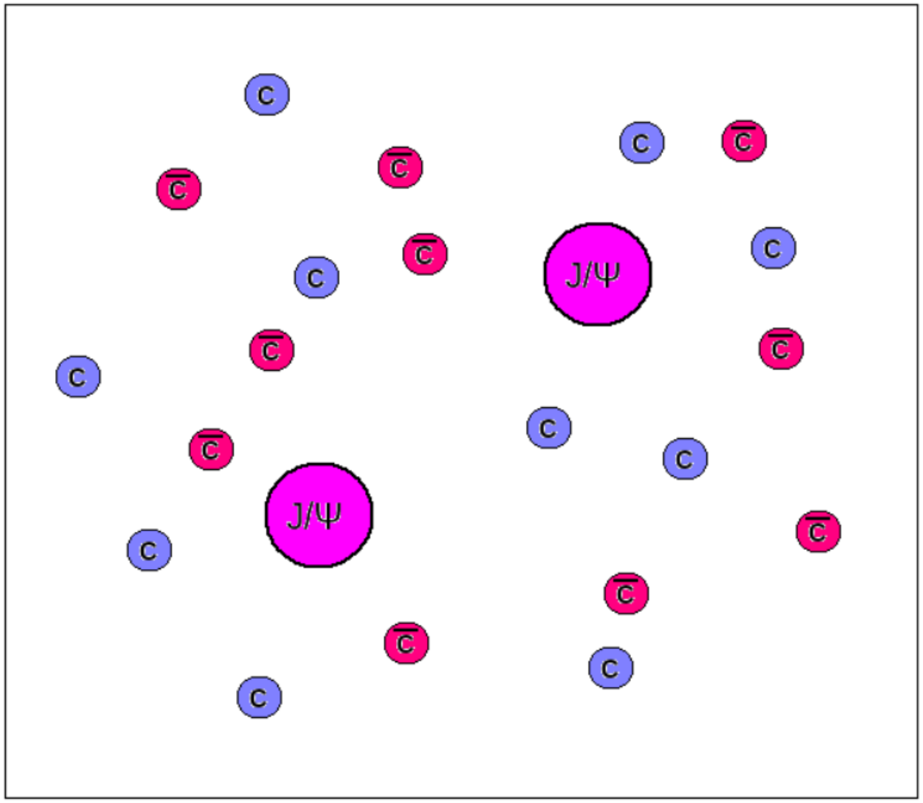

Figure: Simulation setup
(Design: Ian David Ortiz Salcedo)

- Initial charm-anticharm pairs are placed in a box
- Charm quarks move as **Brownian particles** in a thermal medium.
- Charm–anticharm pairs interact via a potential
  - Pair is bound if total relative energy is negative
- **Number of charm and anticharm strictly conserved at any time**

Microscopic simulation vs SHM

Microscopic Simulation

- Fixed number of initial charm pairs
- Charm is conserved **exactly**
- Complies with the **canonical ensemble**

Statistical Hadronization Model

- Framework describing hadron production in high energy collisions
  - Hadrons formed from thermalized QGP
- Multiplicity is derived from **grand canonical ensemble**
- Charm is not conserved exactly, only on **average**

Canonical enhancement

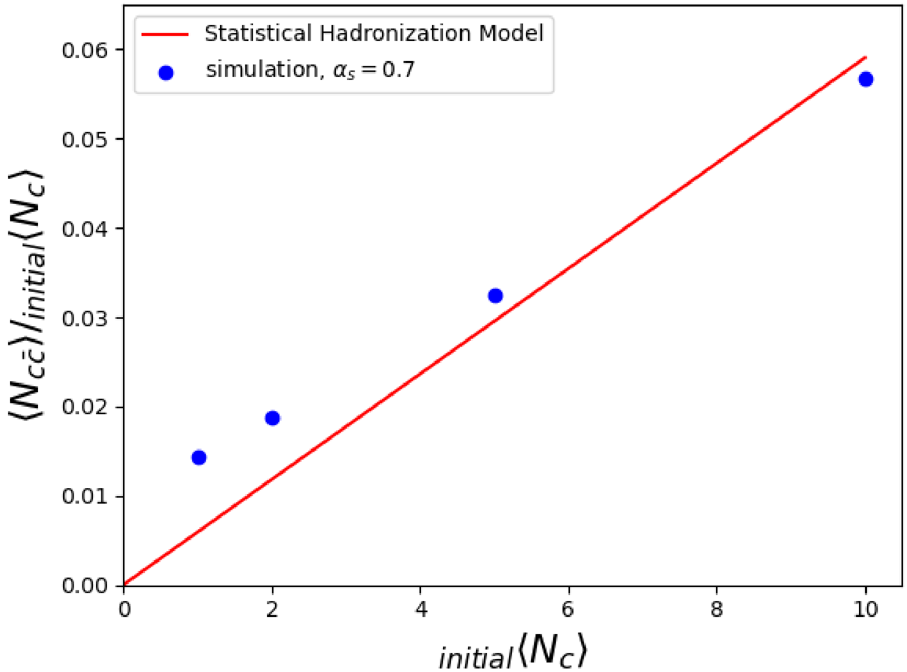

Figure: Canonical enhancement
(data from [PhysRevD.111.074012])

- Canonical approach:
  - Particles **always** occur in pairs
- Grand-canonical approach
  - Particle and antiparticle number only the same on **average**

How does it enhance?

Low number $\langle N_c \rangle$ of initial charm present
Numerical example: $\langle N_c \rangle = 0.1$

Canonical Ensemble	**Grand-Canonical Ensemble**
In **every** event: $N_c = N_{\bar{c}}$	Only on **average**: $\langle N_c \rangle = \langle N_{\bar{c}} \rangle$
$P((N_c \wedge N_{\bar{c}}) \geqslant 1) = P(N_c \geqslant 1) \approx 0.1$	$P(N_{\bar{c}} \geqslant 1) = P(N_c \geqslant 1) \approx 0.1$ $P((N_c \wedge N_{\bar{c}}) \geqslant 1) \approx 0.01$

Master equation for $c\bar{c} \leftrightarrow J/\Psi$

$P_n^{J/\Psi}$ probability for a system with n J/Ψ
P_m^c probability for a system with m c and $\bar{c}$
Transitions are of type $\alpha_{n\to n+1}$, $\alpha_{n\to n-1}$, a particle is **produced** or **decays**

Adapted Master equation

$$\begin{aligned}\frac{\mathrm{d}P_n^{J/\Psi}}{\mathrm{d}t} &= \alpha_{n-1\to n}P_{n-1}^{J/\Psi} + \alpha_{n+1\to n}P_{n+1}^{J/\Psi} - (\alpha_{n\to n+1} + \alpha_{n\to n-1})P_n^{J/\Psi} \\ \frac{\mathrm{d}P_m^{c}}{\mathrm{d}t} &= \alpha_{m-1\to m}P_{m-1}^{c} + \alpha_{m+1\to m}P_{m+1}^{c} - (\alpha_{m\to m+1} + \alpha_{m\to m-1})P_m^{c}\end{aligned} \tag{6}$$

Transition rates

Transition rate of one pair

Γ_G rate for $c\bar{c} \to J/\Psi$ formation $(\alpha_{n\to n+1}, \alpha_{m\to m-1})$
Γ_L rate for $J/\Psi \to c\bar{c}$ decay $(\alpha_{n\to n-1}, \alpha_{m\to m+1})$

$$\Gamma \equiv \frac{\Gamma_G}{\Gamma_L} = \frac{Z^1_{J/\Psi}}{Z^1_c Z^1_{\bar{c}}} \tag{7}$$

Number of possible transitions

For $c\bar{c} \to J/\Psi$: N_c^2 possible combinations.
For $J/\Psi \to c\bar{c}$: $N_{J/\Psi}$ possibilities.

Final Master equation

Master equation for $c\bar{c} \leftrightarrow J/\Psi$

$$\frac{\mathrm{d}P_n^{J/\Psi}}{\mathrm{d}t} = \Gamma_G \left(P_{n-1}^{J/\Psi} - P_n^{J/\Psi}\right)\left(\sum_{m=0}^{\infty} P_m^c m^2\right) - \Gamma_L \left(P_n^{J/\Psi} n - P_{n+1}^{J/\Psi}(n+1)\right)$$

$$\frac{\mathrm{d}P_m^c}{\mathrm{d}t} = \Gamma_L \left(\sum_{n=0}^{\infty} P_n^{J/\Psi} n\right)\left(P_{m-1}^c - P_m^c\right) - \Gamma_G \left(P_m^c m^2 - P_{m+1}^c (m+1)^2\right) \tag{8}$$

Comparing the results

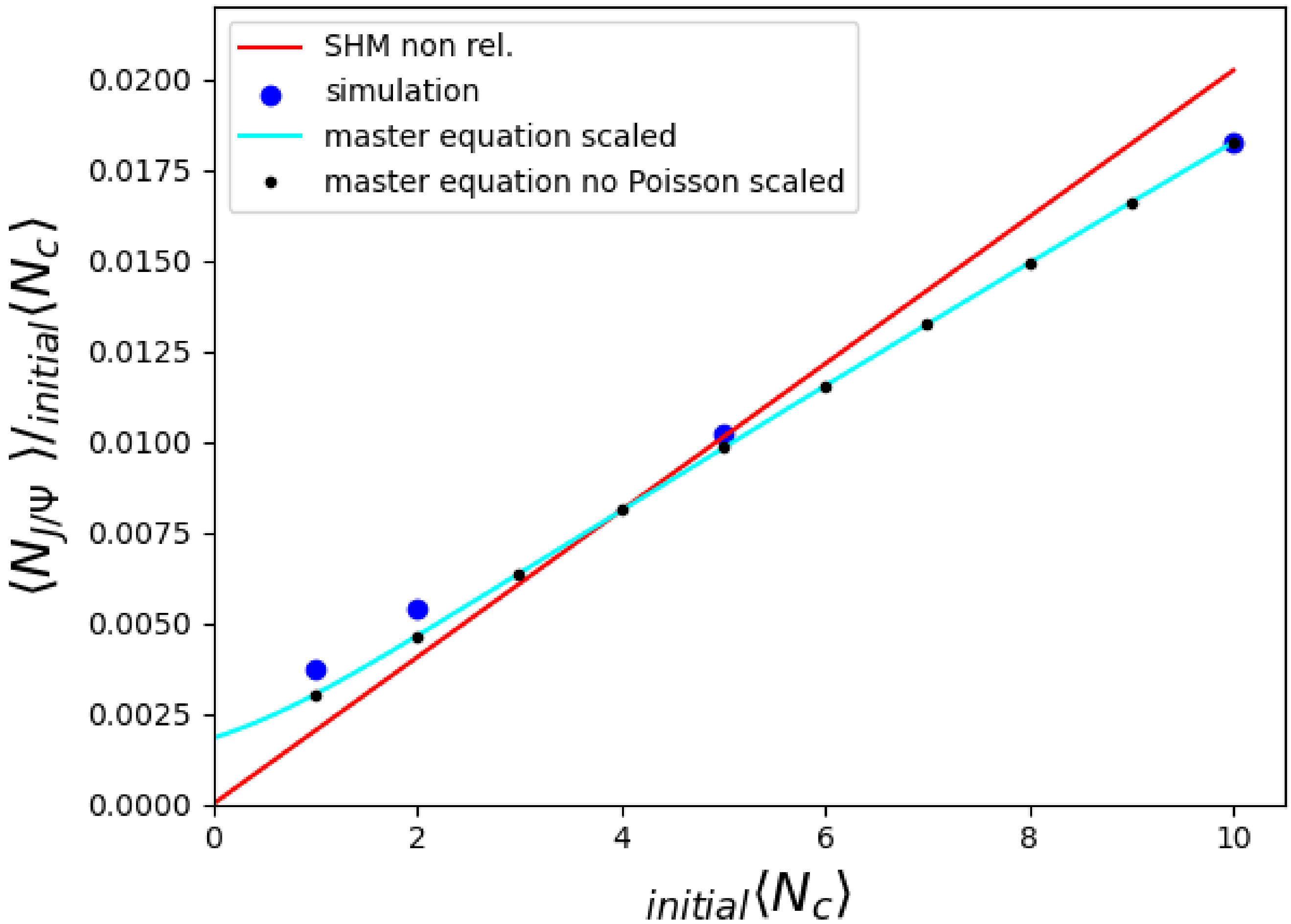

Figure: Microscopic model, the master equations and SHM.

Summary & Outlook

Summary

- Low number of particles: one has to consider canonical effects
  - depending on the setup production is suppressed or enhanced
- Master equations can be used for the canonical approach
  - Seem to align with microscopic simulations

Outlook

- Including other charmonium states
- Other transitions e.g. $b\bar{b} \rightarrow \Upsilon$