Dissipative fluid dynamics in the relaxation-time approximation for an ideal gas of massive particles

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20.06.2024

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V. E. Ambrus, E. Molnár and D. H. Rischke, Phys. Rev. D **106**, no.7, 076005 (2022)

V. E. Ambruș, E. Molnár and D. H. Rischke, Phys. Rev. D **109**, no.7, 076001 (2024)

Introduction and Conclusions I.

Transport coefficients of second-order relativistic fluid dynamics in the relaxation-time approximation

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(Received 12 July 2022; accepted 19 September 2022; published 14 October 2022)

We derive the transport coefficients of second-order fluid dynamics with 14 dynamical moments using the method of moments and the Chapman-Enskog method in the relaxation-time approximation for the collision integral of the relativistic Boltzmann equation. Contrary to results previously reported in the literature, we find that the second-order transport coefficients derived using the two methods are in perfect agreement. Furthermore, we show that, unlike in the case of binary hard-sphere interactions, the diffusion-shear coupling coefficients $\ell_{V\pi}$, $\lambda_{V\pi}$, and $\tau_{V\pi}$ actually diverge in some approximations when the expansion order $N_\ell \to \infty$. Here we show how to circumvent such a problem in multiple ways, recovering the correct transport coefficients of second-order fluid dynamics with 14 dynamical moments. We also validate our results for the diffusion-shear coupling by comparison to a numerical solution of the Boltzmann equation for the propagation of sound waves in an ultrarelativistic ideal gas.

DOI: 10.1103/PhysRevD.106.076005

Introduction and Conclusions II.

Relativistic second-order dissipative and anisotropic fluid dynamics in the relaxation-time approximation for an ideal gas of massive particles

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(Received 3 November 2023; accepted 3 March 2024; published 1 April 2024)

In this paper, we study all transport coefficients of second-order dissipative fluid dynamics derived by V. E. Ambrus *et al.* [Phys. Rev. D **106**, 076005 (2022)] from the relativistic Boltzmann equation in the relaxation-time approximation for the collision integral. These transport coefficients are computed for a classical ideal gas of massive particles, with and without taking into account the conservation of intrinsic quantum numbers. Through rigorous comparison between kinetic theory, second-order dissipative fluid dynamics, and leading-order anisotropic fluid dynamics for a $(0+1)$-dimensional boost-invariant flow scenario, we show that both fluid-dynamical theories describe the early far-from-equilibrium stage of the expansion reasonably well.

DOI: 10.1103/PhysRevD.109.076001

The relativistic Boltzmann equation I.

The Boltzmann equation describes the space-time evolution of the single-particle distribution function $f(x^\mu, k^\mu) = f_{\mathbf{k}}$

The relativistic Boltzmann equation

$$k^\mu \partial_\mu f_{\mathbf{k}} \equiv C[f_{\mathbf{k}}] = \frac{1}{2}\int dK' dP dP' W_{\mathbf{kk}'\to\mathbf{pp}'} \left(f_{\mathbf{p}} f_{\mathbf{p}'} \tilde{f}_{\mathbf{k}} \tilde{f}_{\mathbf{k}'} - f_{\mathbf{k}} f_{\mathbf{k}'} \tilde{f}_{\mathbf{p}} \tilde{f}_{\mathbf{p}'} \right) , \quad (1)$$

$k^\mu = (k^0, \mathbf{k})$ is the four-momenta of particles with mass $m_0 = \sqrt{k^\mu k_\mu}$, where $dK = g d^3\mathbf{k} / \left[(2\pi)^3 k^0\right]$, while $\tilde{f}_{\mathbf{k}} = 1 - a f_{\mathbf{k}}$, with $a = 0/a = 1/a = -1$ for Boltzmann/Fermi/Bose statistics. The invariant transition rate is

$$W_{\mathbf{kk}'\to\mathbf{pp}'} \equiv \frac{s}{g^2}(2\pi)^6 \frac{d\sigma(\sqrt{s},\Omega)}{d\Omega} \delta(k^\mu + k'^\mu - p^\mu - p'^\mu) , \quad (2)$$

which for an isotropic and energy independent diff. cross-section

the hard-sphere approximation

$$\sigma_T \equiv 2\pi \frac{d\sigma(\sqrt{s},\Omega)}{d\Omega} = \frac{1}{n_0 \lambda_{\mathrm{mfp}}} . \quad (3)$$

The relativistic Boltzmann equation II.

Local thermal equilibrium - Jüttner distribution

$$f_{\mathbf{k}} \to f_{0\mathbf{k}} \equiv \left[\exp\left(-\alpha_0 + \beta_0 E_{\mathbf{k}}\right) + a\right]^{-1} , \qquad (4)$$

Not a "true" solution, where $\alpha_0 = \mu_0 / T_0$, $\beta_0 = 1/T_0$ and $E_{\mathbf{k}} = k^{\mu} u_{\mu}$. The solution

$$f_{\mathbf{k}} \equiv f_{0\mathbf{k}} + \delta f_{\mathbf{k}} , \qquad (5)$$

where $\delta f_{\mathbf{k}}$ is the non-equilibrium correction.

The Anderson-Witting approximation to the collision integral

$$C\left[f_{\mathbf{k}}\right] \equiv -k^{\mu} u_{\mu} \frac{f_{\mathbf{k}} - f_{0\mathbf{k}}}{\tau_R} = -E_{\mathbf{k}} \frac{\delta f_{\mathbf{k}}}{\tau_R} , \qquad (6)$$

where the relaxation time $\tau_R(x^{\mu})$ is a momentum-independent parameter proportional to the mean free time between collisions. This is the relativistic generalization of the BGK model; $C_{BGK}\left[f_{\mathbf{k}}\right] \equiv -\frac{f_{\mathbf{k}} - f_{0\mathbf{k}}}{\tau_R}$, used in Lattice Boltzmann methods.

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Fluids dynamics from the Boltzmann equation

The momentum-space integral of the distribution function $f_{\mathbf{k}}$ with the four-momentum and the energy of particles forming a rank-ℓ tensor defines the moment

Rank-ℓ tensor moment

$$M_r^{\mu_1 \cdots \mu_\ell} \equiv \langle E_{\mathbf{k}}^r k^{\mu_1} \cdots k^{\mu_\ell} \rangle = \int dK E_{\mathbf{k}}^r k^{\mu_1} \cdots k^{\mu_\ell} f_{\mathbf{k}} \ . \tag{7}$$

The particle four-current and the energy-momentum tensor

$$N^\mu \equiv M_0^\mu = \int dK k^\mu f_{\mathbf{k}} = \int dK k^\mu \left(f_{0\mathbf{k}} + \delta f_{\mathbf{k}} \right) , \tag{8}$$

$$T^{\mu\nu} \equiv M_0^{\mu\nu} = \int dK k^\mu k^\nu f_{\mathbf{k}} = \int dK k^\mu k^\nu \left(f_{0\mathbf{k}} + \delta f_{\mathbf{k}} \right) . \tag{9}$$

Moments of the relativistic Boltzmann equation (RTA)

$$\int dK k^{\lambda_1} \cdots k^{\lambda_\ell} \left[k^\mu \partial_\mu f_{\mathbf{k}} \right] \equiv \partial_\mu M_0^{\mu\lambda_1 \cdots \lambda_\ell} = C^{\lambda_1 \cdots \lambda_\ell} \left[f_{\mathbf{k}} \right] , \tag{10}$$

$$= -\frac{1}{\tau_R} \left(M_1^{\lambda_1 \cdots \lambda_\ell} - \int dK E_{\mathbf{k}} k^{\lambda_1} \cdots k^{\lambda_\ell} f_{0\mathbf{k}} \right)$$

Note that the particle number and particle four-momenta are collision invariants, i.e., $C = \int dK C[f] = 0$ and $C^\alpha = \int dK k^\alpha C[f] = 0$

The particle number conservation

$$\partial_\mu N^\mu \equiv \partial_\mu \int dK k^\mu f_{\mathbf{k}} = -\frac{1}{\tau_R} \int dK E_{\mathbf{k}} \left(f_{\mathbf{k}} - f_{0\mathbf{k}} \right)$$
$$= -\frac{1}{\tau_R} \left(n - n_0 \right) \equiv 0 \,, \tag{11}$$

The energy-momentum conservation

$$u_\nu \partial_\mu T^{\mu\nu} \equiv u_\nu \partial_\mu \int dK k^\mu k^\nu f_{\mathbf{k}} = -\frac{1}{\tau_R} \int dK E_{\mathbf{k}}^2 \left(f_{\mathbf{k}} - f_{0\mathbf{k}} \right)$$
$$= -\frac{1}{\tau_R} \left(e - e_0 \right) \equiv 0 \,, \tag{12}$$
$$\Delta^\lambda_\nu \partial_\mu T^{\mu\nu} \equiv \Delta^\lambda_\nu \partial_\mu \int dK k^\mu k^\nu f_{\mathbf{k}} = -\frac{1}{\tau_R} \int dK E_{\mathbf{k}} k^{\langle\lambda\rangle} \left(f_{\mathbf{k}} - f_{0\mathbf{k}} \right)$$
$$= -\frac{1}{\tau_R} W^\lambda \equiv 0 \,, \tag{13}$$

Landau matching conditions, and Landau frame

$$n = n_0 \,, \qquad e = e_0 \,, \tag{14}$$
$$W^\lambda = 0 \,, \tag{15}$$

Ideal Fluids I.

Conservation laws for a simple (single component) perfect fluid (no dissipation)

$$\partial_\mu N_0^\mu = 0 \quad \textbf{charge conservation} \quad \Rightarrow \textbf{1 eq.}$$
$$\partial_\mu T_0^{\mu\nu} = 0 \quad \textbf{energy-momentum conservation} \quad \Rightarrow \textbf{4 eqs.}$$
$$u_\nu \partial_\mu T_0^{\mu\nu} = 0, \qquad \Delta^\lambda_\nu \partial_\mu T_0^{\mu\nu} = 0$$

Perfect fluid decomposition with respect to u^μ

$$N_0^\mu = n_0 u^\mu$$
$$T_0^{\mu\nu} = e_0 u^\mu u^\nu - p_0 \Delta^{\mu\nu}$$
$$n_0 = N_0^\mu u_\mu \quad \textbf{(net)charge density}$$
$$e_0 = T_0^{\mu\nu} u_\mu u_\nu \quad \textbf{energy density}$$
$$P_0 = -\frac{1}{3}\Delta_{\mu\nu} T_0^{\mu\nu} \quad \textbf{equilibrium pressure}$$

- The time-like normalized flow velocity is $u^\mu(t, \vec{x}) = \gamma(1, \mathbf{v})$, where $u^\mu u_\mu = 1$
- The projection tensor $\Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu$, where $g^{\mu\nu} = diag(1, -1, -1, -1)$
- We have 5 equations for 6 unknowns $n_0(1), e_0(1), p_0(1)$ and $u^\mu(3)$. Closed by an Equation of State (EoS) $P_0 = P_0(e_0, n_0)$.

Dissipative Fluids I.

Conservation laws for a simple (single component) dissipative fluid

$\partial_\mu N^\mu = 0$ **charge conservation** $\Rightarrow$ **1 eq.**

$\partial_\mu T^{\mu\nu} = 0$ **energy-momentum conservation** $\Rightarrow$ **4 eqs.**

General decomposition $N^\mu = N_0^\mu + \delta N^\mu$ and $T^{\mu\nu} = T_0^{\mu\nu} + \delta T^{\mu\nu}$

$$N^\mu = n u^\mu + V^\mu$$

$$T^{\mu\nu} = e u^\mu u^\nu - (P_0 + \Pi)\Delta^{\mu\nu} + W^\mu u^\nu + W^\nu u^\mu + \pi^{\mu\nu}$$

$n = N^\mu u_\mu$ **charge density**

$e = T^{\mu\nu} u_\mu u_\nu$ **energy density**

$P \equiv P_0 + \Pi \equiv -\frac{1}{3}\Delta_{\mu\nu} T^{\mu\nu}$ **isotropic pressure**

$V^\mu = \Delta^{\mu\alpha} N_\alpha$ **charge flow**

$W^\mu = \Delta^{\mu\alpha} u^\beta T_{\alpha\beta}$ **energy-momentum flow**

$\pi^{\mu\nu} = \left[\frac{1}{2}\left(\Delta^{\mu\alpha}\Delta^{\nu\beta} + \Delta^{\mu\beta}\Delta^{\alpha\nu}\right) - \frac{1}{3}\Delta^{\mu\nu}\Delta^{\alpha\beta}\right] T_{\alpha\beta}$ **stress tensor**

- We only have 5 equations and an EoS for 17 unknowns, $n(1), e(1), u^\mu(3)$ and $\Pi(1), V^\mu(3), W^\mu(3), \pi^{\mu\nu}(5)$.

General equations of motion I.

Using $f_{\mathbf{k}} = f_{0\mathbf{k}} + \delta f_{\mathbf{k}}$ where $\delta f_{\mathbf{k}} = f_{0\mathbf{k}} \left(1 - a f_{0\mathbf{k}}\right) \phi_{\mathbf{k}}$ we define the **irreducible moment**

$$\rho_r^{\mu_1 \cdots \mu_\ell} \equiv \left\langle E_{\mathbf{k}u}^r k^{\langle \mu_1} \cdots k^{\mu_\ell \rangle} \right\rangle_\delta \tag{16}$$

$\langle \cdots \rangle_\delta \equiv \langle \cdots \rangle - \langle \cdots \rangle_0 = \int dK (\cdots) \delta f_{\mathbf{k}}$ and $k^{\langle \mu_1} \cdots k^{\mu_\ell \rangle} = \Delta^{\mu_1 \cdots \mu_\ell}_{\nu_1 \cdots \nu_\ell} k^{\nu_1} \cdots k^{\nu_\ell}$

The primary (14) dynamical moments in N^μ and $T^{\mu\nu}$

$$\rho_1 \equiv \delta n = 0 \,, \qquad \rho_2 \equiv \delta e = 0 \,, \qquad \rho_0 \equiv -\frac{3}{m^2} \Pi \,,$$
$$\rho_0^\mu \equiv V^\mu \,, \qquad \rho_1^\mu \equiv W^\mu = 0 \,, \qquad \rho_0^{\mu\nu} \equiv \pi^{\mu\nu} \,.$$

Now, writing the Boltzmann equation in the following form

$$D\delta f_{\mathbf{k}} = -D f_{0\mathbf{k}} - E_{\mathbf{k}u}^{-1} k_\nu \nabla^\nu \left(f_{0\mathbf{k}} + \delta f_{\mathbf{k}}\right) + E_{\mathbf{k}u}^{-1} C \left[f_{0\mathbf{k}} + \delta f_{\mathbf{k}}\right] \,, \tag{17}$$

where $D \equiv u^\mu \partial_\mu = \frac{d}{d\tau}$ and $\nabla_\mu = \Delta_\mu^\nu \partial_\nu$, the equations for $D\rho_r^{\mu_1 \cdots \mu_\ell}$ follow from,

$$D\rho_r^{\langle \mu_1 \cdots \mu_\ell \rangle} \equiv \Delta^{\mu_1 \cdots \mu_\ell}_{\nu_1 \cdots \nu_\ell} \frac{d}{d\tau} \int dK E_{\mathbf{k}u}^r k^{\langle \nu_1} \cdots k^{\nu_\ell \rangle} \delta f_{\mathbf{k}} \,. \tag{18}$$

General equations of motion II.

Infinitely many coupled equations for $\rho_r^{\mu_1\cdots\mu_\ell}$ equivalent to the Boltzmann equation !

Scalar, vector, and tensor equations

$$\dot{\rho}_r - C_{r-1} = \alpha_r^{(0)}\theta + \frac{\theta}{3}\left[m_0^2(r-1)\rho_{r-2} - (r+2)\rho_r - 3\frac{G_{2r}}{D_{20}}\Pi\right]$$
$$+ \frac{G_{3r}}{D_{20}}\partial_\mu V^\mu - \nabla_\mu \rho_{r-1}^\mu + r\rho_{r-1}^\mu \dot{u}_\mu + \left[(r-1)\rho_{r-2}^{\mu\nu} + \frac{G_{2r}}{D_{20}}\pi^{\mu\nu}\right]\sigma_{\mu\nu} ,$$

$$\dot{\rho}_r^{\langle\mu\rangle} - C_{r-1}^{\langle\mu\rangle} = \alpha_r^{(1)}\nabla^\mu\alpha + r\rho_{r-1}^{\mu\nu}\dot{u}_\nu - \frac{1}{3}\nabla^\mu\left(m_0^2\rho_{r-1} - \rho_{r+1}\right) - \Delta_\alpha^\mu\left(\nabla_\nu\rho_{r-1}^{\alpha\nu} + \alpha_r^h\partial_\kappa\pi^{\kappa\alpha}\right)$$
$$+ \frac{1}{3}\left[m_0^2\left(r-1\right)\rho_{r-2}^\mu - \left(r+3\right)\rho_r^\mu\right]\theta + \frac{1}{5}\sigma^{\mu\nu}\left[2m_0^2\left(r-1\right)\rho_{r-2,\nu} - \left(2r+3\right)\rho_{r,\nu}\right]$$
$$+ \frac{1}{3}\left[m_0^2 r\rho_{r-1} - \left(r+3\right)\rho_{r+1} - 3\alpha_r^h\Pi\right]\dot{u}^\mu + \alpha_r^h\nabla^\mu\Pi + \rho_{r,\nu}\omega^{\mu\nu} + \left(r-1\right)\rho_{r-2}^{\mu\nu\lambda}\sigma_{\nu\lambda} ,$$

$$\dot{\rho}_r^{\langle\mu\nu\rangle} - C_{r-1}^{\langle\mu\nu\rangle} = 2\alpha_r^{(2)}\sigma^{\mu\nu} + \frac{2}{15}\left[m_0^4\left(r-1\right)\rho_{r-2} - m_0^2\left(2r+3\right)\rho_r + \left(r+4\right)\rho_{r+2}\right]\sigma^{\mu\nu} + 2\rho_r^{\lambda\langle\mu}\omega^{\nu\rangle}{}_\lambda$$
$$+ \frac{2}{5}\dot{u}^{\langle\mu}\left[m_0^2 r\rho_{r-1}^{\nu\rangle} - \left(r+5\right)\rho_{r+1}^{\nu\rangle}\right] - \frac{2}{5}\nabla^{\langle\mu}\left(m_0^2\rho_{r-1}^{\nu\rangle} - \rho_{r+1}^{\nu\rangle}\right)$$
$$+ \frac{1}{3}\left[m_0^2\left(r-1\right)\rho_{r-2}^{\mu\nu} - \left(r+4\right)\rho_r^{\mu\nu}\right]\theta + \frac{2}{7}\left[2m_0^2\left(r-1\right)\rho_{r-2}^{\lambda\langle\mu} - \left(2r+5\right)\rho_r^{\lambda\langle\mu}\right]\sigma_\lambda^{\nu\rangle}$$
$$+ r\rho_{r-1}^{\mu\nu\gamma}\dot{u}_\gamma - \Delta_{\alpha\beta}^{\mu\nu}\nabla_\lambda\rho_{r-1}^{\alpha\beta\lambda} + \left(r-1\right)\rho_{r-2}^{\mu\nu\lambda\kappa}\sigma_{\lambda\kappa} . \qquad (19)$$

G. S. Denicol, H. Niemi, E. Molnar and D. H. Rischke, "Derivation of transient relativistic fluid dynamics from the Boltzmann equation," Phys. Rev. D **85**, 114047 (2012)

General equations of motion III.

The Dirk Not MotheR (DNMR) recipe:

Expansion around equilibrium

$$f_{\mathbf{k}} = f_{0\mathbf{k}} + f_{0\mathbf{k}} \left(1 - a f_{0\mathbf{k}}\right) \sum_{\ell=0}^{\infty} \sum_{n=0}^{N_\ell} \rho_n^{\mu_1 \cdots \mu_\ell} k_{\langle \mu_1} \cdots k_{\mu_\ell \rangle} \mathcal{H}_{\mathbf{k}n}^{(\ell)} , \tag{20}$$

$$\mathcal{H}_{\mathbf{k}n}^{(\ell)} = \frac{(-1)^\ell}{\ell! \; J_{2\ell,\ell}} \sum_{i=n}^{N_\ell} \sum_{m=0}^{i} a_{in}^{(\ell)} a_{im}^{(\ell)} E_{\mathbf{k}}^m . \tag{21}$$

The coefficients $a_{ij}^{(\ell)}$ are calculated via the Gram-Schmidt orthogonalization procedure

Define the negative-order moments $r < 0$

$$\rho_{\pm r}^{\mu_1 \cdots \mu_\ell} = \sum_{n=0}^{N_\ell} \rho_n^{\mu_1 \cdots \mu_\ell} \mathcal{F}_{\mp r,n}^{(\ell)} , \tag{22}$$

$$\mathcal{F}_{\mp rn}^{(\ell)} \equiv \frac{\ell!}{(2\ell+1)!!} \int \mathrm{d}K \, E_{\mathbf{k}}^{\pm r} \left(\Delta^{\alpha\beta} k_\alpha k_\beta\right)^\ell \mathcal{H}_{\mathbf{k}n}^{(\ell)} f_{0\mathbf{k}} \left(1 - a f_{0\mathbf{k}}\right) . \tag{23}$$

The Grad or Israel-Stewart approximations have no negative-moments!

General equations of motion IV.

The moments of the linearized collision integral

$$C_{r-1}^{\langle\mu_1\cdots\mu_\ell\rangle} \equiv -\sum_{n=0}^{N_\ell} \mathcal{A}_{rn}^{(\ell)} \rho_n^{\mu_1\cdots\mu_\ell} , \qquad \sum_{r=0}^{N_\ell} \tau_{nr}^{(\ell)} C_{r-1}^{\langle\mu_1\cdots\mu_\ell\rangle} = -\rho_n^{\langle\mu_1\cdots\mu_\ell\rangle} , \tag{24}$$

where $\mathcal{A}_{in}^{(\ell)}$ is the collision matrix while its inverse

$$\tau_{rn}^{(\ell)} \equiv \left(\mathcal{A}^{(\ell)}\right)_{rn}^{-1} = \sum_{m=0}^{N_\ell} \Omega_{rm}^{(\ell)} \frac{1}{\chi_m^{(\ell)}} \left(\Omega^{(\ell)}\right)_{mn}^{-1} , \tag{25}$$

is the microscopic time scales proportional to $\tau_{\rm mfp} = \lambda_{\rm mfp}/c$ between collisions. Here $\Omega^{(\ell)}$ diagonalizes $\mathcal{A}^{(\ell)}$ according to; $\left(\Omega^{(\ell)}\right)^{-1} \mathcal{A}^{(\ell)}\Omega^{(\ell)} = {\rm diag}\left(\chi_0^{(\ell)}, \chi_1^{(\ell)}, \cdots\right)$ while we set $\Omega_{00}^{(\ell)} = 1$, while arrange the eigenvalues in increasing order, $\chi_r^{(\ell)} \leq \chi_{r+1}^{(\ell)}$.

The RTA collision integral

$$C_{r-1}^{\langle\mu_1\cdots\mu_\ell\rangle} \equiv -\sum_{n=0}^{N_\ell} \frac{\delta_{rn}^{(\ell)}}{\tau_R} \rho_n^{\mu_1\cdots\mu_\ell} = -\frac{1}{\tau_R}\rho_r^{\mu_1\cdots\mu_\ell} , \quad \tau_{rn}^{(\ell)} = \tau_R \delta_{rn} , \quad \Omega_{rn}^{(\ell)} = \delta_{rn} . \tag{26}$$

General equations of motion V.

The moment equations

$$\dot{\rho}_r + \sum_{n=0}^{N_0} \mathcal{A}_{rn}^{(0)} \rho_n = \alpha_r^{(0)} \theta + (\text{higher-order terms}) \, , \tag{27}$$

$$\dot{\rho}_r^{\langle\mu\rangle} + \sum_{n=0}^{N_1} \mathcal{A}_{rn}^{(1)} \rho_n^{\mu} = \alpha_r^{(1)} \nabla^{\mu} \alpha + (\text{higher-order terms}) \, , \tag{28}$$

$$\dot{\rho}_r^{\langle\mu\nu\rangle} + \sum_{n=0}^{N_2} \mathcal{A}_{rn}^{(2)} \rho_n^{\mu\nu} = 2\alpha_r^{(2)} \sigma^{\mu\nu} + (\text{higher-order terms}) \, , \tag{29}$$

The moment equations - final form

$$\sum_{r=0}^{N_0} \tau_{nr}^{(0)} \dot{\rho}_r + \rho_n = \theta \sum_{r=0}^{N_0} \tau_{nr}^{(0)} \alpha_r^{(0)} + \sum_{r=0}^{N_0} \tau_{nr}^{(0)} (\text{higher-order terms}) \, , \tag{30}$$

$$\sum_{r=0}^{N_1} \tau_{nr}^{(1)} \dot{\rho}_r^{\langle\mu\rangle} + \rho_n^{\mu} = \nabla^{\mu} \alpha \sum_{r=0}^{N_1} \tau_{nr}^{(1)} \alpha_r^{(1)} + \sum_{r=0}^{N_1} \tau_{nr}^{(1)} (\text{higher-order terms}) \, , \tag{31}$$

$$\sum_{r=0}^{N_2} \tau_{nr}^{(2)} \dot{\rho}_r^{\langle\mu\nu\rangle} + \rho_n^{\mu\nu} = 2\sigma^{\mu\nu} \sum_{r=0}^{N_2} \tau_{nr}^{(2)} \alpha_r^{(2)} + \sum_{r=0}^{N_2} \tau_{nr}^{(2)} (\text{higher-order terms}) \, , \tag{32}$$

General equations of motion VI.

The 14 dynamical moments are $\rho_0 = -\frac{3}{m_0^2}\Pi$, $\rho_0^\mu = V^\mu$, and $\rho_0^{\mu\nu} = \pi^{\mu\nu}$,

The DNMR approximation for the non-dynamical moments

$$\rho_{r>0} \simeq -\frac{3}{m_0^2}\Omega_{r0}^{(0)}\Pi + \frac{3}{m_0^2}(\zeta_r - \Omega_{r0}^{(0)}\zeta_0)\theta\,, \qquad \rho_{-r} \simeq -\frac{3}{m_0^2}\gamma_{r0}^{(0)}\Pi + O(Kn)\,, \tag{33}$$

$$\rho_{r>0}^\mu \simeq \Omega_{r0}^{(1)}V^\mu + (\kappa_r - \Omega_{r0}^{(1)}\kappa_0)\nabla^\mu\alpha\,, \qquad \rho_{-r}^\mu \simeq \gamma_{r0}^{(1)}V^\mu + O(Kn)\,, \tag{34}$$

$$\rho_{r>0}^{\mu\nu} \simeq \Omega_{r0}^{(2)}\pi^{\mu\nu} + 2(\eta_r - \Omega_{r0}^{(2)}\eta_0)\sigma^{\mu\nu}\,, \qquad \rho_{-r}^{\mu\nu} \simeq \gamma_{r0}^{(2)}\pi^{\mu\nu} + O(Kn). \tag{35}$$

where the first-order transport coefficients and the $\gamma_{r0}^{(\ell)}$ coefficients are:

$$\zeta_r \equiv \frac{m_0^2}{3}\sum_{n=0,\neq 1,2}^{N_0}\tau_{rn}^{(0)}\alpha_n^{(0)}\,, \quad \kappa_r \equiv \sum_{n=0,\neq 1}^{N_1}\tau_{rn}^{(1)}\alpha_n^{(1)}\,, \quad \eta_r \equiv \sum_{n=0}^{N_2}\tau_{rn}^{(2)}\alpha_n^{(2)}\,, \tag{36}$$

$$\gamma_{r0}^{(0)} \equiv \sum_{n=0,\neq 1,2}^{N_0}\mathcal{F}_{rn}^{(0)}\Omega_{n0}^{(0)}\,, \quad \gamma_{r0}^{(1)} \equiv \sum_{n=0,\neq 1}^{N_1}\mathcal{F}_{rn}^{(1)}\Omega_{n0}^{(1)}\,, \quad \gamma_{r0}^{(2)} \equiv \sum_{n=0}^{N_2}\mathcal{F}_{rn}^{(2)}\Omega_{n0}^{(2)}\,. \tag{37}$$

For $r = 0$ one obtains the usual Navier-Stokes coefficients: $\zeta_0 \equiv \tau_{00}^{(0)}\alpha_0^{(0)} = \tau_\Pi\alpha_0^{(0)}$, $\kappa_0 = \tau_V\alpha_0^{(1)}$, and $\eta_0 = \tau_\pi\alpha_0^{(2)}$.

General equations of motion VII.

The corrected DNMR approximation for the non-dynamical moments

$$\rho_{-r} \simeq -\frac{3}{m_0^2}\gamma_{r0}^{(0)}\Pi + \frac{3}{m_0^2}(\Gamma_{r0}^{(0)} - \gamma_{r0}^{(0)})\xi_0\theta \quad \Longrightarrow \quad \rho_{-r} \simeq \frac{3}{m_0^2}\Gamma_{r0}^{(0)}, \tag{38}$$

$$\rho_{-r}^{\mu} \simeq \gamma_{r0}^{(1)}V^{\mu} + (\Gamma_{r0}^{(1)} - \gamma_{r0}^{(1)})\kappa_0\nabla^{\mu}\alpha \quad \Longrightarrow \quad \rho_{-r}^{\mu} \simeq \Gamma_{r0}^{(1)}V^{\mu}, \tag{39}$$

$$\rho_{-r}^{\mu\nu} \simeq \gamma_{r0}^{(2)}\pi^{\mu\nu} + 2(\Gamma_{r0}^{(2)} - \gamma_{r0}^{(2)})\eta_0\sigma^{\mu\nu} \quad \Longrightarrow \quad \rho_{-r}^{\mu\nu} \simeq \Gamma_{r0}^{(2)}\pi^{\mu\nu}. \tag{40}$$

where the $O(Kn)$ corrections are now explicitly included, $cDNMR$

$$\gamma_{r0}^{(0)} \equiv \sum_{n=0,\neq 1,2}^{N_0} \mathcal{F}_{rn}^{(0)}\Omega_{n0}^{(0)}, \quad \gamma_{r0}^{(1)} \equiv \sum_{n=0,\neq 1}^{N_1} \mathcal{F}_{rn}^{(1)}\Omega_{n0}^{(1)}, \quad \gamma_{r0}^{(2)} \equiv \sum_{n=0}^{N_2} \mathcal{F}_{rn}^{(2)}\Omega_{n0}^{(2)}, \tag{41}$$

$$\Gamma_{r0}^{(0)} \equiv \sum_{n=0,\neq 1,2}^{N_0} \mathcal{F}_{rn}^{(0)}\frac{\zeta_n}{\zeta_0}, \quad \Gamma_{r0}^{(1)} \equiv \sum_{n=0,\neq 1}^{N_1} \mathcal{F}_{rn}^{(1)}\frac{\kappa_n}{\kappa_0}, \quad \Gamma_{r0}^{(2)} \equiv \sum_{n=0}^{N_2} \mathcal{F}_{rn}^{(2)}\frac{\eta_n}{\eta_0}. \tag{42}$$

V. E. Ambrus, E. Molnár and D. H. Rischke, "Transport coefficients of second-order relativistic fluid dynamics in the relaxation-time approximation," Phys. Rev. D **106**, no.7, 076005 (2022)

General equations of motion VIII.

This is the way in RTA!

$$\tau_R \dot{\rho}_r + \rho_r = \tau_R \alpha_r^{(0)} \theta + O(2) = \frac{3}{m_0^2} \xi_r \theta + O(\mathrm{Re}^{-1}\mathrm{Kn}) \tag{43}$$

$$\tau_R \dot{\rho}_r^{\langle\mu\rangle} + \rho_r^{\langle\mu\rangle} = \tau_R \alpha_r^{(1)} \nabla^\mu \alpha + O(2) = \kappa_r \nabla^\mu \alpha + O(\mathrm{Re}^{-1}\mathrm{Kn}) \,, \tag{44}$$

$$\tau_R \dot{\rho}_r^{\langle\mu\nu\rangle} + \rho_r^{\langle\mu\nu\rangle} = 2\tau_R \alpha_r^{(2)} \sigma^{\mu\nu} + O(2) = 2\eta_r \sigma^{\mu\nu} + O(\mathrm{Re}^{-1}\mathrm{Kn}) \,, \tag{45}$$

where $\zeta_r = \tau_R \frac{m_0^2}{3} \alpha_r^{(0)}$, $\kappa_r = \tau_R \alpha_r^{(1)}$ and $\eta_r = \tau_R \alpha_r^{(2)}$, hence the ratio of coefficients,

$$\mathcal{R}_{r0}^{(\ell)} \equiv \frac{\alpha_r^{(\ell)}}{\alpha_0^{(\ell)}} \Longrightarrow \quad \mathcal{R}_{r0}^{(0)} = \frac{\zeta_r}{\zeta_0} \,, \quad \mathcal{R}_{r0}^{(1)} = \frac{\kappa_r}{\kappa_0} \,, \quad \mathcal{R}_{r0}^{(2)} = \frac{\eta_r}{\eta_0} \,. \tag{46}$$

The order of things in RTA!

$$\rho_{r\neq 0} \simeq -\frac{3}{m_0^2} \mathcal{R}_{r0}^{(0)} \Pi \,, \quad \rho_{r\neq 0}^{\mu} \simeq \mathcal{R}_{r0}^{(1)} V^\mu \,, \quad \rho_{r\neq 0}^{\mu\nu} \simeq \mathcal{R}_{r0}^{(2)} \pi^{\mu\nu} \,, \tag{47}$$

V. E. Ambrus, E. Molnár and D. H. Rischke, "Transport coefficients of second-order relativistic fluid dynamics in the relaxation-time approximation," Phys. Rev. D **106**, no.7, 076005 (2022)

Second-order fluid dynamics I.

Equation for the bulk pressure

$$\tau_\Pi \dot{\Pi} + \Pi = -\zeta\theta - \ell_{\Pi V}\nabla_\mu V^\mu - \tau_{\Pi V} V_\mu \dot{u}^\mu - \delta_{\Pi\Pi}\Pi\theta - \lambda_{\Pi V} V_\mu \nabla^\mu \alpha + \lambda_{\Pi\pi}\pi^{\mu\nu}\sigma_{\mu\nu} \,, \qquad (48)$$

Equation for the diffusion current

$$\tau_V \dot{V}^{\langle\mu\rangle} + V^\mu = \kappa\nabla^\mu\alpha + -\tau_V V_\nu \omega^{\nu\mu} - \delta_{VV} V^\mu\theta - \ell_{V\Pi}\nabla^\mu\Pi + \ell_{V\pi}\Delta^{\mu\nu}\nabla_\lambda \pi^\lambda{}_\nu + \tau_{V\Pi}\Pi\dot{u}^\mu - \tau_{V\pi}\pi^{\mu\nu}\dot{u}_\nu - \lambda_{VV} V_\nu \sigma^{\mu\nu} + \lambda_{V\Pi}\Pi\nabla^\mu\alpha - \lambda_{V\pi}\pi^{\mu\nu}\nabla_\nu\alpha \,, \qquad (49)$$

Equation for the shear-stress tensor

$$\tau_\pi \dot{\pi}^{\langle\mu\nu\rangle} + \pi^{\mu\nu} = 2\eta\sigma^{\mu\nu} + 2\tau_\pi \pi_\lambda^{\langle\mu}\omega^{\nu\rangle\lambda} - \delta_{\pi\pi}\pi^{\mu\nu}\theta - \tau_{\pi\pi}\pi^{\lambda\langle\mu}\sigma_\lambda^{\nu\rangle} + \lambda_{\pi\Pi}\Pi\sigma^{\mu\nu} - \tau_{\pi V} V^{\langle\mu}\dot{u}^{\nu\rangle} + \ell_{\pi V}\nabla^{\langle\mu}V^{\nu\rangle} + \lambda_{\pi V} V^{\langle\mu}\nabla^{\nu\rangle}\alpha \,. \qquad (50)$$

First-order transport coefficients

$$\zeta = \tau_\Pi \frac{m_0^2}{3}\alpha_0^{(0)} \,, \quad \kappa = \tau_V \alpha_0^{(1)} \,, \quad \eta = \tau_\pi \alpha_0^{(2)} \,. \qquad (51)$$

Second-order fluid dynamics II.

The second-order coefficients in the 14-moment approximation are:

$$\delta_{\Pi\Pi} = \tau_\Pi \left(\frac{2}{3} - \frac{m_0^2}{3} \frac{G_{20}}{D_{20}} + \frac{m_0^2}{3} \mathcal{R}^{(0)}_{-2,0} \right), \quad \lambda_{\Pi V} = -\tau_\Pi \frac{m_0^2}{3} \left(\frac{\partial \mathcal{R}^{(1)}_{-1,0}}{\partial \alpha} + \frac{1}{h} \frac{\partial \mathcal{R}^{(1)}_{-1,0}}{\partial \beta} \right),$$

$$\ell_{\Pi V} = \tau_\Pi \frac{m_0^2}{3} \left(\frac{G_{30}}{D_{20}} - \mathcal{R}^{(1)}_{-1,0} \right), \tau_{\Pi V} = -\tau_\Pi \frac{m_0^2}{3} \left(\frac{G_{30}}{D_{20}} - \frac{\partial \mathcal{R}^{(1)}_{-1,0}}{\partial \ln \beta} \right), \lambda_{\Pi \pi} = -\tau_\Pi \frac{m_0^2}{3} \left(\frac{G_{20}}{D_{20}} - \mathcal{R}^{(2)}_{-2,0} \right).$$

$$\delta_{VV} = \tau_V \left(1 + \frac{m_0^2}{3} \mathcal{R}^{(1)}_{-2,0} \right), \quad \ell_{V\Pi} = \frac{\tau_V}{h} \left(1 - h \mathcal{R}^{(0)}_{-1,0} \right), \quad \ell_{V\pi} = \frac{\tau_V}{h} \left(1 - h \mathcal{R}^{(2)}_{-1,0} \right),$$

$$\tau_{V\Pi} = \frac{\tau_V}{h} \left(1 - h \frac{\partial \mathcal{R}^{(0)}_{-1,0}}{\partial \ln \beta} \right), \quad \tau_{V\pi} = \frac{\tau_V}{h} \left(1 - h \frac{\partial \mathcal{R}^{(2)}_{-1,0}}{\partial \ln \beta} \right), \quad \lambda_{VV} = \tau_V \left(\frac{3}{5} + \frac{2m_0^2}{5} \mathcal{R}^{(1)}_{-2,0} \right),$$

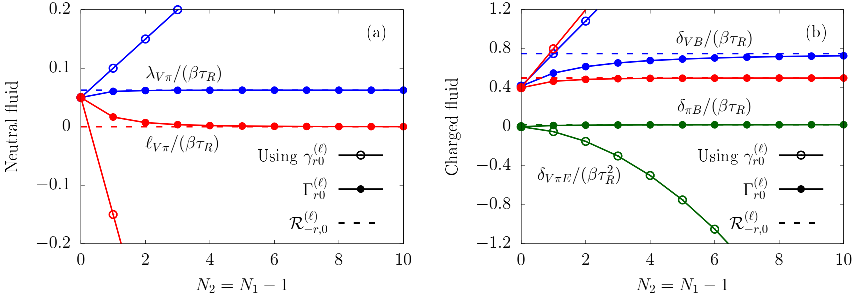
$$\lambda_{V\Pi} = \tau_V \left(\frac{\partial \mathcal{R}^{(0)}_{-1,0}}{\partial \alpha} + \frac{1}{h} \frac{\partial \mathcal{R}^{(0)}_{-1,0}}{\partial \beta} \right), \quad \lambda_{V\pi} = \tau_V \left(\frac{\partial \mathcal{R}^{(2)}_{-1,0}}{\partial \alpha} + \frac{1}{h} \frac{\partial \mathcal{R}^{(2)}_{-1,0}}{\partial \beta} \right).$$

$$\delta_{\pi\pi} = \tau_\pi \left(\frac{4}{3} + \frac{m_0^2}{3} \mathcal{R}^{(2)}_{-2,0} \right), \quad \tau_{\pi\pi} = \tau_\pi \left(\frac{10}{7} + \frac{4m_0^2}{7} \mathcal{R}^{(2)}_{-2,0} \right), \quad \lambda_{\pi\Pi} = \tau_\pi \left(\frac{6}{5} + \frac{2m_0^2}{5} \mathcal{R}^{(0)}_{-2,0} \right),$$

$$\tau_{\pi V} = -\tau_\pi \frac{2m_0^2}{5} \frac{\partial \mathcal{R}^{(1)}_{-1,0}}{\partial \ln \beta}, \quad \ell_{\pi V} = -\tau_\pi \frac{2m_0^2}{5} \mathcal{R}^{(1)}_{-1,0}, \quad \lambda_{\pi V} = -\tau_\pi \frac{2m_0^2}{5} \left(\frac{\partial \mathcal{R}^{(1)}_{-1,0}}{\partial \alpha} + \frac{1}{h} \frac{\partial \mathcal{R}^{(1)}_{-1,0}}{\partial \beta} \right).$$

Second-order fluid dynamics III.

Coefficients in the massless limit



All other coefficients agree in all approximations; i.e., γ_{r0} vs. Γ_{r0} vs. $\mathcal{R}_{-r,0}$.

Second-order fluid dynamics IV.

With a binary collision integral

For a gas of ultrarelativistic hard spheres with constant cross-section, the 14-dynamical moment approximation for truncation orders $N_0 - 2 = N_1 - 1 = N_2 = 1000$, corresponding to $N_0 + 3N_1 + 5N_2 + 9 = 9014$ moments included in the basis.

Coefficients of the diffusion equation

Method	κ	$\tau_V[\lambda_{\rm mfp}]$	$\delta_{VV}[\tau_V]$	$\ell_{V\pi}[\tau_V] = \tau_{V\pi}[\tau_V]$	$\lambda_{VV}[\tau_V]$	$\lambda_{V\pi}[\tau_V]$
14M	$1/12\sigma$	$9/4$	1	$\beta/20$	$3/5$	$\beta/20$
IReD	$0.15892/\sigma$	2.0838	1	0.028371β	0.89862	0.069273β
DNMR	$0.15892/\sigma$	2.5721	1	0.11921β	0.92095	0.051709β
cDNMR	$0.15892/\sigma$	2.5721	1	0.098534β	0.92095	0.056878β

Coefficients of the shear-stress equation

Method	η	$\tau_\pi[\lambda_{\rm mfp}]$	$\delta_{\pi\pi}[\tau_\pi]$	$\ell_{\pi V}[\tau_\pi]$	$\tau_{\pi V}[\tau_\pi]$	$\tau_{\pi\pi}[\tau_\pi]$	$\lambda_{\pi V}[\tau_\pi]$
14M	$4/(3\sigma\beta)$	$5/3$	$4/3$	0	0	$10/7$	0
IReD	$1.2676/(\sigma\beta)$	1.6557	$4/3$	$-0.56960/\beta$	$-2.2784/\beta$	1.6945	$0.20503/\beta$
c&DNMR	$1.2676/(\sigma\beta)$	2	$4/3$	$-0.68317/\beta$	$-2.7327/\beta$	1.6888	$0.24188/\beta$

D. Wagner, V. E. Ambrus and E. Molnar, "Analytical structure of the binary collision integral and the ultrarelativistic limit of transport coefficients of an ideal gas," Phys. Rev. D **109**, no.5, 056018 (2024)

Applications I.

Relativistic ideal gas of massive particles

Relativistic ideal gas I.

Thermodynamic integrals

$$I_{rq}(\alpha,\beta) \equiv \frac{(-1)^r}{(2q+1)!!}\int dK E_{\mathbf{k}}^{r-2q}\left(\Delta^{\mu\nu}k_\mu k_\nu\right)^q f_{0\mathbf{k}}$$
$$= \frac{g e^\alpha}{2\pi^2}\frac{m_0^{r+2}}{(2q+1)!!}\int_1^\infty \mathrm{d}x\, x^{r-2q}(x^2-1)^{q+\frac{1}{2}} e^{-zx}\,, \qquad (52)$$

where $z = m_0/T$ and $f_{0\mathbf{k}} = e^{\alpha-\beta E_{\mathbf{k}}}$, while

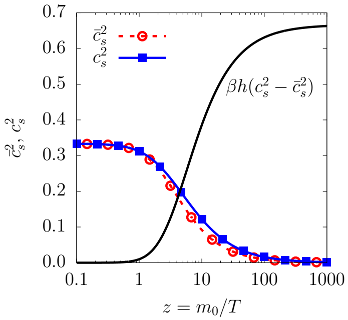
Bessel functions of second-kind

$$K_q(z) \equiv \frac{z^q}{(2q-1)!!}\int_1^\infty \mathrm{d}x\,(x^2-1)^{q-\frac{1}{2}} e^{-zx}\,. \qquad (53)$$

Primary thermodynamic quantities

$$n \equiv I_{10} = \frac{g e^\alpha}{2\pi^2} T^3 z^2 K_2(z)\,, \quad P \equiv I_{21} = nT\,, \quad e \equiv I_{20} = P\left[3 + z\frac{K_1(z)}{K_2(z)}\right]\,. \qquad (54)$$

Relativistic ideal gas II.



The speed of sound - with N^μ conservation

$$c_s^2 \equiv \left(\frac{\partial P}{\partial e}\right)_n + \frac{1}{h}\left(\frac{\partial P}{\partial n}\right)_e = \frac{c_p P}{c_v(e+P)}\,, \qquad (55)$$

The speed of sound - withOUT N^μ conservation

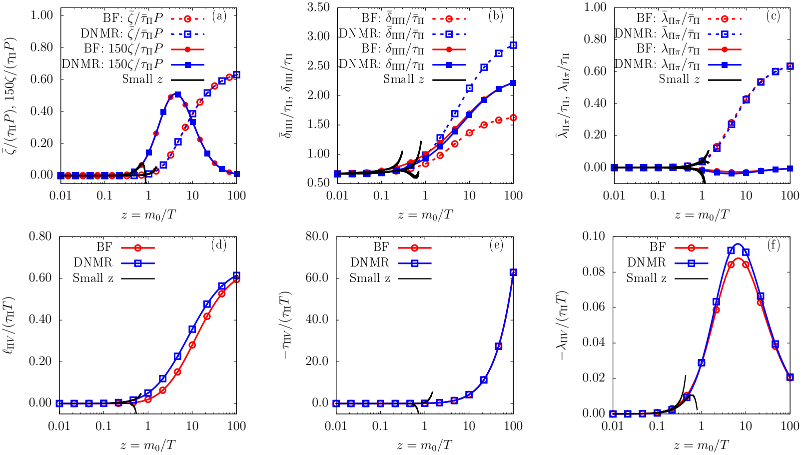
$$\bar{c}_s^2 \equiv \left(\frac{\partial P}{\partial e}\right)_\mu = \frac{I_{31}}{I_{30}} = \frac{P(e+P)}{c_v P^2 + e^2}\,, \qquad (56)$$

where c_v and $c_p = c_v + 1$ are heat capacities at constant volume or pressure.

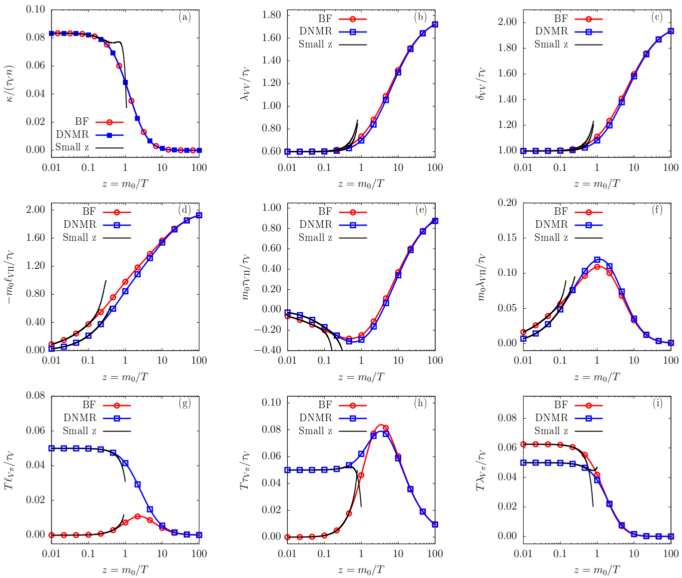
The coefficient(s) of bulk viscosity

$$\frac{\zeta}{\tau_\Pi} = \frac{5}{3}\frac{\eta}{\tau_\pi} - c_s^2\,(e+P)\,, \quad \frac{\bar{\zeta}}{\tau_\Pi} = \frac{5}{3}\frac{\eta}{\tau_\pi} - \bar{c}_s^2\,(e+P)\,.$$

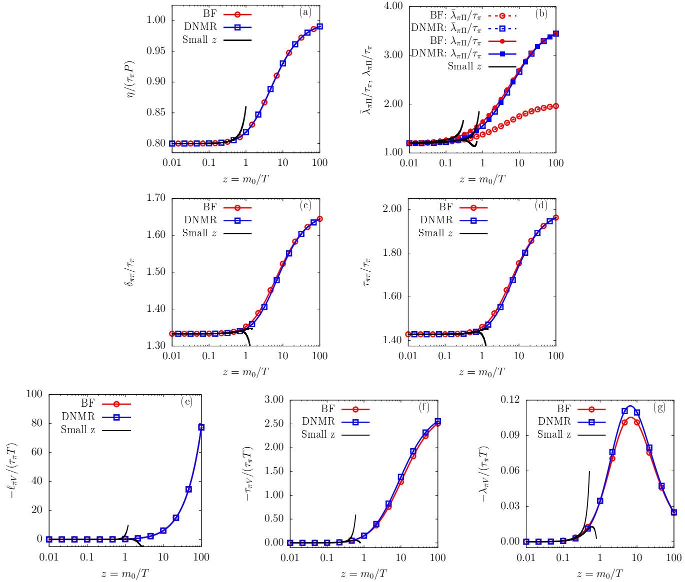
Coefficients of the bulk equation



Coefficients of the diffusion equation



Coefficients of the shear equation



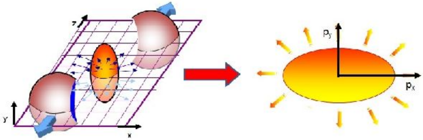
Applications II.

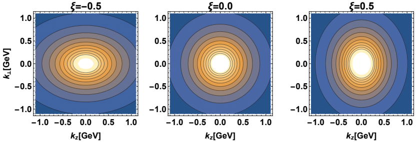
Bjorken expansion

$$u^\mu \equiv \left(\frac{t}{\tau}, 0, 0, \frac{z}{\tau}\right) = (\cosh\eta_s, 0, 0, \sinh\eta_s)$$

$$l^\mu \equiv \left(\frac{z}{\tau}, 0, 0, \frac{t}{\tau}\right) = (\sinh\eta_s, 0, 0, \cosh\eta_s)$$

Applications II.





- Elliptic flow - momentum space anisotropy of particle emission in non-central heavy-ion collisions.

$$\hat{f}_{RS} \equiv \left[\exp\left(-\beta\sqrt{k_\perp^2 + (1+\xi)k_z^2}\right)\right]$$

Spheroidal momentum distribution function. Prolate (left) - ISOtropic - Oblate (right)

Anisotropic fluid dynamics

$$\hat{N}^\mu = \hat{n}u^\mu + \hat{n}_l l^\mu \tag{57}$$

$$\hat{T}^{\mu\nu} = \hat{e}u^\mu u^\nu + 2\hat{M}u^{(\mu}l^{\nu)} + \hat{P}_l l^\mu l^\nu - \hat{P}_\perp \Xi^{\mu\nu} \tag{58}$$

where $\Xi^{\mu\nu} \equiv g^{\mu\nu} - u^\mu u^\nu + l^\mu l^\nu = \Delta^{\mu\nu} + l^\mu l^\nu$ such that $\Xi^{\mu\nu} u_n u = \Xi^{\mu\nu} l_\nu = 0$.

Boltzmann equation for the boost invariant expansion

$$\frac{\partial f_{\mathbf{k}}}{\partial \tau} - \frac{v^z}{\tau}(1 - v_z^2)\frac{\partial f_{\mathbf{k}}}{\partial v_z} = -\frac{1}{\tau_R}(f_{\mathbf{k}} - f_{0\mathbf{k}}) , \qquad v^z \equiv \tanh(y - \eta_s)$$

Second-order fluid dynamics - 4 diff equations

$$Dn + \frac{n}{\tau} = 0 , \qquad De + \frac{1}{\tau}(e + P_l) = 0 , \qquad P_l \equiv P + \Pi - \pi , \quad P_\perp \equiv P + \Pi + \frac{\pi}{2} ,$$

$$\tau_R D\Pi + \Pi = -\frac{\zeta}{\tau} - \delta_{\Pi\Pi}\frac{\Pi}{\tau} + \lambda_{\Pi\pi}\frac{\pi}{\tau} , \qquad \tau_R D\pi + \pi = \frac{4\eta}{3\tau} - \delta_{\pi\pi}\frac{\pi}{\tau} - \tau_{\pi\pi}\frac{\pi}{3\tau} + \lambda_{\pi\Pi}\frac{2\Pi}{3\tau} .$$

Anisotropic fluid dynamics - 3 diff equations

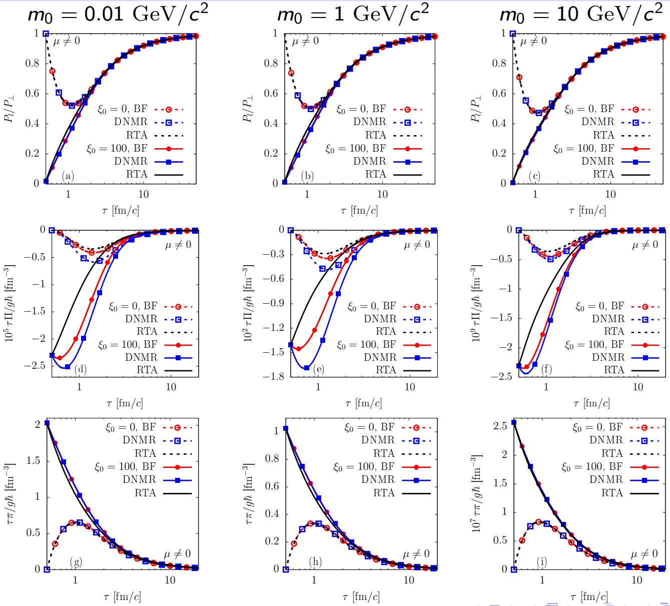
$$\hat{f}_{RS} = \exp\left(\hat{\alpha} - \frac{k^\tau}{\Lambda}\sqrt{1 + \xi v_z^2}\right) , \qquad \hat{I}^{RS}_{000} = \int dK \hat{f}_{RS} , \qquad \hat{I}^{RS}_{240} = \int dK\, E_{\mathbf{k}u}^{-2} E_{\mathbf{k}l}^{4} \hat{f}_{RS} ,$$

$$D\hat{n} + \frac{\hat{n}}{\tau} = -\frac{1}{\tau_R}(\hat{n} - n) , \qquad D\hat{e} + \frac{1}{\tau}\left(\hat{e} + \hat{P}_l\right) = -\frac{1}{\tau_R}(\hat{e} - e) , \qquad \hat{P}_\perp \equiv \hat{I}^{RS}_{201} = \frac{1}{2}\left(e - \hat{P}_l - m_0^2 \hat{I}^{RS}_{000}\right)$$

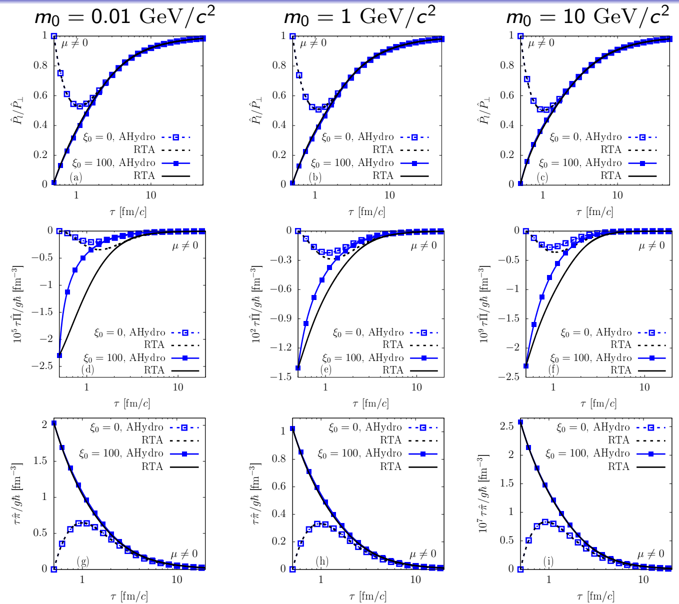
$$D\hat{P}_l + \frac{1}{\tau}\left(3\hat{P}_l - \hat{I}^{RS}_{240}\right) = -\frac{1}{\tau_R}\left(\hat{P}_l - P\right) , \qquad \hat{\Pi} = \frac{1}{3}\left(\hat{P}_l + 2\hat{P}_\perp\right) - P , \quad \hat{\pi} = \frac{2}{3}(\hat{P}_\perp - \hat{P}_l) .$$

There are no free parameters, everything above is a function of $z = m_0/T$.

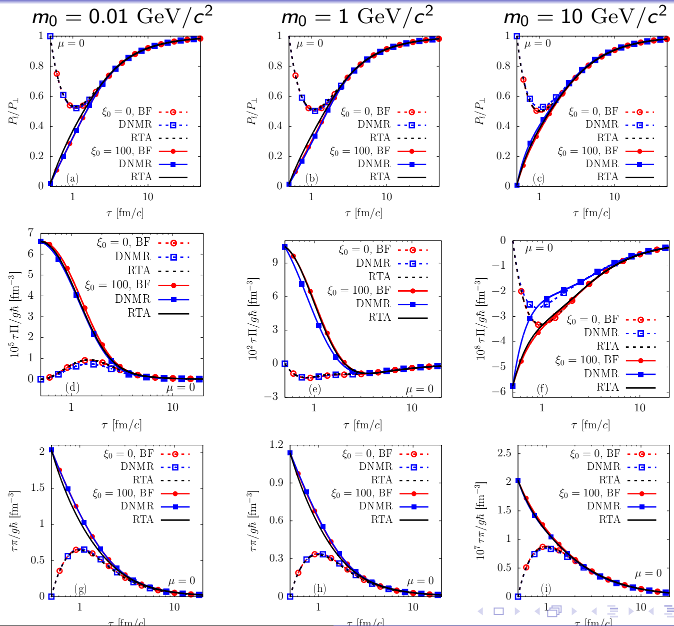
Kinetic theory vs. second-order fluid dynamics - WITH conserved particles



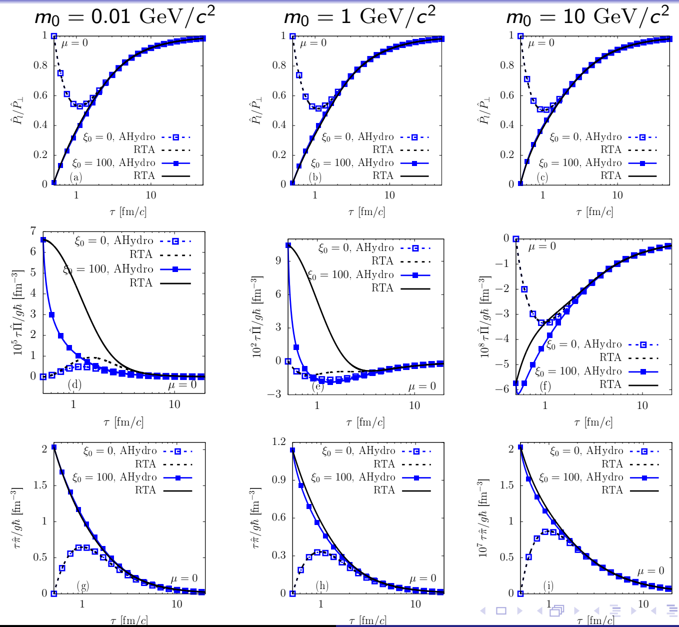
Kinetic theory vs. anisotropic fluid dynamics - WITH conserved particles



Kinetic theory vs. second-order fluid dynamics - NO conserved particles



Kinetic theory vs. anisotropic fluid dynamics - NO conserved particles



The End