Relativistic hydrodynamics with spin

Enrico Speranza

W. Florkowski, B. Friman, A. Jaiswal, ES, Phys. Rev. C **97**, 041901 (2018)
W. Florkowski, B. Friman, A. Jaiswal, R. Ryblewski, ES, arXiv:1712.07676
W. Florkowski, ES, F. Becattini, arXiv:1803.11098

Transport Meeting
Goethe University

Frankfurt, May 3rd, 2018

Outline

- Relativistic perfect-fluid dynamics with spin degrees of freedom based on distribution functions

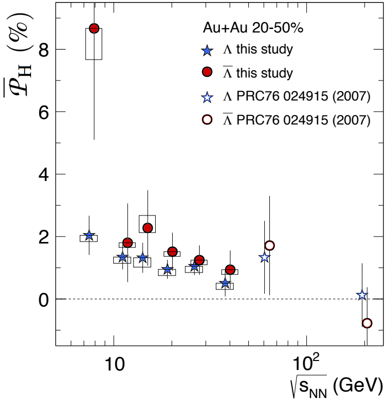
  W. Florkowski, B. Friman, A. Jaiswal, ES, Phys. Rev. C **97**, 041901 (2018)

- Study formal aspects of distribution functions
  - Connection between spin-polarization 3-vector, spin tensor and Pauli-Lubański 4-vector

  W. Florkowski, B. Friman, A. Jaiswal, R. Ryblewski, ES, arXiv:1712.07676

Motivation

- Non-central nuclear collisions ⇒ Large global angular momentum
  ⇒ May generate spin polarization of hot and dense matter
- Connection between spin polarization and vorticity
- Measurement of Λ hyperon polarization: "Most vortical fluid"



L. Adamczyk et al. (STAR), Nature 548 62-65
H. Petersen, Nature (News & Views)

Basic conservation laws

Poincare symmetry leads to basic conservation laws

- Energy-momentum tensor $T^{\mu\nu}$
  Conservation of energy and momentum:

$$\partial_\mu T^{\mu\nu} = 0$$

- Total angular momentum tensor ("orbital"+"spin")

$$J^{\lambda,\mu\nu} = x^\mu T^{\lambda\nu} - x^\nu T^{\lambda\mu} + S^{\lambda,\mu\nu}$$

Conservation of total angular momentum:

$$\partial_\lambda J^{\lambda,\mu\nu} = 0 \Longrightarrow \partial_\lambda S^{\lambda,\mu\nu} = T^{\nu\mu} - T^{\mu\nu}$$

Spin tensor $S^{\lambda,\mu\nu}$ is in general not conserved

Global thermodynamic equilibrium

- Density operator for quantum mechanical system

$$\rho = \exp\left[-\int d^3\Sigma_\mu(x)\left(T^{\mu\nu}(x)\beta_\nu - \frac{1}{2}S^{\mu,\alpha\beta}(x)\omega_{\alpha\beta}\right)\right]$$

Global thermodynamic equilibrium

$$\Downarrow$$

ρ independent of time

$$\Downarrow$$

$$\partial_\mu\beta_\nu + \partial_\nu\beta_\mu = 0,$$

$$\omega_{\mu\nu} = -\frac{1}{2}\left(\partial_\mu\beta_\nu - \partial_\nu\beta_\mu\right) = \text{const} \quad \text{(thermal vorticity)}$$

Global vs. local equilibrium

Present phenomenology prescription used to describe the data:

1) Run any type of hydro (perfect or viscous)
2) Find $\beta_\mu(x) = u_\mu(x)/T(x)$ on the freeze-out hypersurface
3) Calculate thermal vorticity
4) Identify thermal vorticity with the spin-polarization tensor $\omega_{\mu\nu}$
5) Make predictions about spin polarization

This talk:

In **local equilibrium** thermal vorticity and $\omega_{\mu\nu}$ are in general different

$$\omega_{\mu\nu} \neq -\frac{1}{2}\left(\partial_\mu \beta_\nu - \partial_\nu \beta_\mu\right)$$

$\beta_\mu(x)$ and $\omega_{\mu\nu}(x)$ are independent quantities

Spin polarization may be early-stage effect that survives the whole evolution

Local distribution functions for spin-1/2 particles

- Starting point (Becattini et al., Annals. Phys. 338 32):

$$f_{rs}^{+}(x,p) = \frac{1}{2m}\bar{u}_r(p)X^{+}u_s(p), \qquad f_{rs}^{-}(x,p) = -\frac{1}{2m}\bar{v}_s(p)X^{-}v_r(p)$$

$$X^{\pm} = \exp\left[\pm\xi(x) - \beta_\mu(x)p^\mu\right] M^{\pm}$$

$$M^{\pm} = \exp\left[\pm\frac{1}{2}\omega_{\mu\nu}(x)\Sigma^{\mu\nu}\right]$$

with $\beta^\mu = u^\mu/T$, $\xi = \mu/T$, $\Sigma^{\mu\nu} = (i/4)[\gamma^\mu,\gamma^\nu]$

- $\omega_{\mu\nu}$ analogue to EM field-strength tensor $F_{\mu\nu} = E_\mu u_\nu - E_\nu u_\mu + \epsilon_{\mu\nu\beta\gamma}u^\beta B^\gamma$

$$\omega_{\mu\nu} \equiv k_\mu u_\nu - k_\nu u_\mu + \epsilon_{\mu\nu\beta\gamma}u^\beta\omega^\gamma$$

- Polarization tensor expressed like EM field-strength tensor

$$\omega_{\mu\nu} = \begin{bmatrix} 0 & e^1 & e^2 & e^3 \\ -e^1 & 0 & -b^3 & b^2 \\ -e^2 & b^3 & 0 & -b^1 \\ -e^3 & -b^2 & b^1 & 0 \end{bmatrix}$$

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Spin matrices $M^{\pm}$

- General expression

$$M^{\pm} = 1_{4\times 4}\left[\Re(\cosh z) \pm \Re\left(\frac{\sinh z}{2z}\right)\omega_{\mu\nu}\Sigma^{\mu\nu}\right] + i\gamma_5\left[\Im(\cosh z) \pm \Im\left(\frac{\sinh z}{2z}\right)\omega_{\mu\nu}\Sigma^{\mu\nu}\right]$$

with $z = \frac{1}{2\sqrt{2}}\sqrt{\omega_{\mu\nu}\omega^{\mu\nu} + i\omega_{\mu\nu}\tilde{\omega}^{\mu\nu}} = \frac{1}{2}\sqrt{k\cdot k - \omega\cdot\omega + 2ik\cdot\omega}$

- Assumptions: $k\cdot\omega = 0$, and $k\cdot k - \omega\cdot\omega \geq 0$ (ζ is real)

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- ζ imaginary – W. Florkowski, ES, F. Becattini, arXiv:1803.11098

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Charge current and energy-momentum tensor

- **Charge current** (de Groot, van Leeuwen, van Weert)

$$N^\mu = \int \frac{d^3p}{2(2\pi)^3 E_p} p^\mu \left[\mathrm{tr}_4(X^+) - \mathrm{tr}_4(X^-)\right] = n u^\mu$$

$$n = 4\cosh(\zeta)\sinh(\xi)\, n_{(0)}(T) = \underbrace{\left(e^\zeta + e^{-\zeta}\right)}_{\text{spin-up + spin-down}} \overbrace{\left(e^\xi - e^{-\xi}\right)}^{\text{particles − antiparticles}} n_{(0)}(T)$$

Boltzmann average

$$n_{(0)}(T) \equiv \int \frac{d^3p}{(2\pi)^3 E_p} (u\cdot p)\, e^{-\beta\cdot p}$$

- **Energy-momentum tensor** (de Groot, van Leeuwen, van Weert)

$$T^{\mu\nu} = \int \frac{d^3p}{2(2\pi)^3 E_p} p^\mu p^\nu \left[\mathrm{tr}_4(X^+) + \mathrm{tr}_4(X^-)\right] = (\varepsilon + P) u^\mu u^\nu - P g^{\mu\nu},$$

$$\varepsilon = 4\cosh(\zeta)\cosh(\xi)\, \varepsilon_{(0)}(T)$$
$$P = 4\cosh(\zeta)\cosh(\xi)\, P_{(0)}(T),$$

Entropy current

Generalization of the Boltzmann expression

$$S^\mu = -\int \frac{d^3p}{2(2\pi)^3 E_p}\, p^\mu \left(\mathrm{tr}_4 \left[X^+ (\ln X^+ - 1) \right] + \mathrm{tr}_4 \left[X^- (\ln X^- - 1) \right] \right)$$

$$s = u_\mu S^\mu = \frac{\varepsilon + P - \mu\, n - \Omega w}{T}$$

Ω defined through the relation $\zeta = \Omega / T$

$$w = 4\,\sinh(\zeta)\cosh(\xi)\, n_{(0)}$$

New thermodynamic variable Ω – "Spin chemical potential"

$$s = \left. \frac{\partial P(T,\mu,\Omega)}{\partial T} \right|_{\mu,\Omega}, \quad n = \left. \frac{\partial P(T,\mu,\Omega)}{\partial \mu} \right|_{T,\Omega}, \quad w = \left. \frac{\partial P(T,\mu,\Omega)}{\partial \Omega} \right|_{T,\mu}$$

Hydrodynamic spin background equations

- Conservation of energy and momentum:

$$\partial_\mu T^{\mu\nu} = 0 \quad \Longrightarrow \quad \partial_\mu[(\varepsilon + P)u^\mu] = u^\mu \partial_\mu P \equiv \frac{dP}{d\tau}$$

Evaluating the derivatives

$$T\,\partial_\mu(su^\mu) + \mu\,\partial_\mu(nu^\mu) + \Omega\,\partial_\mu(wu^\mu) = 0.$$

Charge conservation:

$$\partial_\mu(nu^\mu) = 0.$$

- Perfect fluid $\Rightarrow$ entropy conservation $\Rightarrow$ we demand:

Additional conservation law

$$\partial_\mu(wu^\mu) = 0$$

Self-consistent entropy conservation $\partial_\mu(su^\mu) \; = \; 0$

Spin dynamics

Symmetric $T^{\mu\nu}$

$$\Downarrow$$

$$\partial_\lambda S^{\lambda,\mu\nu} = 0$$

Form of the spin tensor (Becattini, Tinti, Annals Phys. 325, 1566)

$$S^{\lambda,\mu\nu} = \int \frac{d^3p}{2(2\pi)^3 E_p}\, p^\lambda \, \mathrm{tr}_4 \left[(X^+ - X^-) \Sigma^{\mu\nu} \right] = \frac{w u^\lambda}{4\zeta} \omega^{\mu\nu}$$

Rescaled spin-polarization tensor $\bar{\omega}^{\mu\nu} = \omega^{\mu\nu}/(2\zeta)$

$$u^\lambda \partial_\lambda \bar{\omega}^{\mu\nu} = \frac{d\bar{\omega}^{\mu\nu}}{d\tau} = 0$$

- Parallel transport of the spin polarization direction along the fluid stream lines
- Non-trivial spin dynamics

Global equilibrium with rotation
Stationary vortex (I)

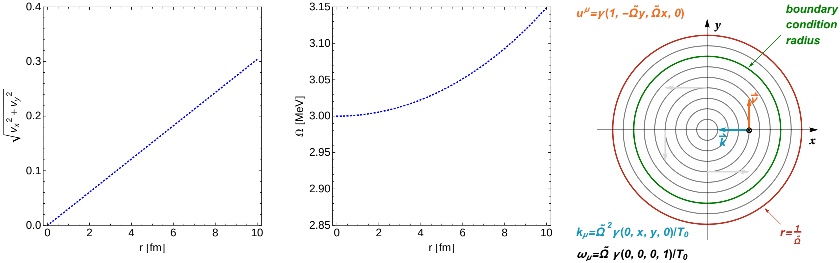
- Hydrodynamic flow $u^\mu = \gamma(1, \boldsymbol{v})$ (rigid rotor):

$$u^0 = \gamma, \quad u^1 = -\gamma\,\tilde{\Omega}\,y, \quad u^2 = \gamma\,\tilde{\Omega}\,x, \quad u^3 = 0,$$

  $\tilde{\Omega}$ is a constant, $\gamma = 1/\sqrt{1-\tilde{\Omega}^2 r^2}$, $r^2 = x^2 + y^2$
  Due to limiting light speed, $0 \leq r \leq R < 1/\tilde{\Omega}$

- Solution for hydrodynamic spin background:

$$T = T_0\gamma, \quad \mu = \mu_0\gamma, \quad \Omega = \Omega_0\gamma$$



Global equilibrium with rotation

Stationary vortex (II)

- Unpolarized vortex: $\omega_{\mu\nu} = 0$ and $\Omega_0 = 0$
- **Polarized** vortex (global equilibrium):

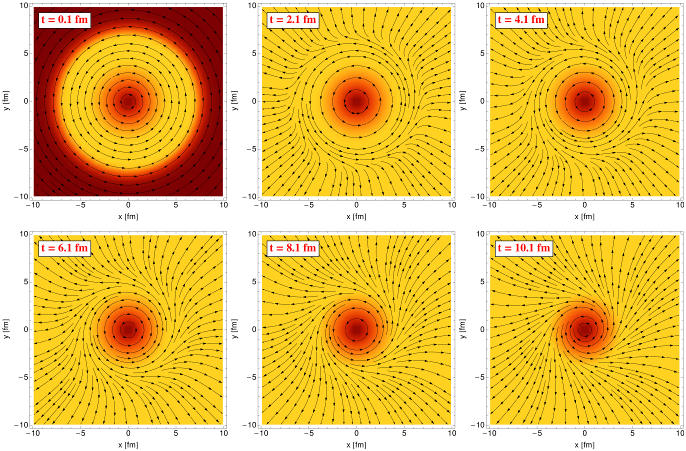
$$\omega_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \tilde{\Omega}/T_0 & 0 \\ 0 & -\tilde{\Omega}/T_0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \tilde{\Omega} = 2\,\Omega_0.$$

Spin-polarization tensor = thermal vorticity:

$$\omega_{\mu\nu} = -\frac{1}{2}\left(\partial_\mu \beta_\nu - \partial_\nu \beta_\mu\right)$$

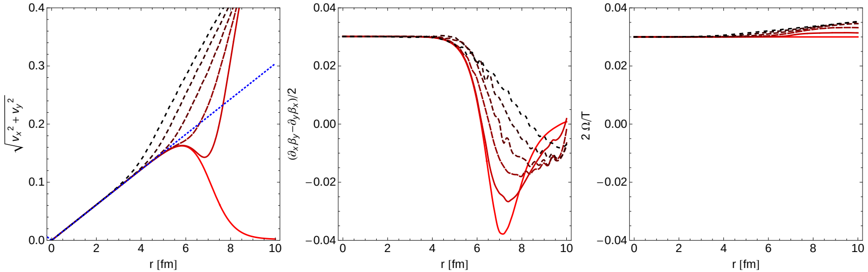
Isolated vortex (I)

- External boundary is removed $\Longrightarrow$ Expansion into external vacuum
- Time dependent problem solved numerically



Color gradient: strength of fluid velocity

Isolated vortex (II)



Time increases by 2 fm: red → black lines

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Spin-density matrix

- **Pure state**: $|\psi\rangle = \sum_\lambda c_\lambda |\lambda\rangle$
  Expectation value of an operator $\langle O\rangle = \langle\psi|O|\psi\rangle$
- **Mixed state**: incoherent mixture of $|\psi_i\rangle$ with statistical weight a_i

$$f = \sum_i a_i |\psi_i\rangle\langle\psi_i| = \sum_{\lambda,\lambda'} f_{\lambda\lambda'} |\lambda\rangle\langle\lambda'|$$

$f_{\lambda\lambda'} = \sum_i a_i\, c_\lambda^{(i)} c_{\lambda'}^{(i)*}$. Expectation value: $\langle O\rangle = \mathrm{Tr}(f\, O)$

Spin-1/2 particle (2×2 hermitian matrix):

$$f = \frac{1}{2}(1 + \mathcal{P}\cdot\boldsymbol{\sigma})$$

- Polarization 3-vector: $\mathcal{P} = \langle\boldsymbol{\sigma}\rangle = \mathrm{Tr}(f\,\boldsymbol{\sigma})$

$|\mathcal{P}| = 1$ Pure state

$0 < |\mathcal{P}| < 1$ Mixed state

$|\mathcal{P}| = 0$ Completely unpolarized mixed state

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Polarization 3-vector $\mathcal{P}$

- Expansion in terms of Pauli matrices

$$f^{\pm}(x,p) = e^{\pm\xi - p\cdot\beta}\left[\cosh(\zeta) - \frac{\sinh(\zeta)}{2\zeta}\,\boldsymbol{P}\cdot\boldsymbol{\sigma}\right]$$

$$\boldsymbol{P} = \frac{1}{m}\left[E_p\,\boldsymbol{b} - \boldsymbol{p}\times\boldsymbol{e} - \frac{\boldsymbol{p}\cdot\boldsymbol{b}}{E_p+m}\boldsymbol{p}\right] = \boldsymbol{b}_*$$

$*$ denotes the PARTICLE REST FRAME

- Average polarization vector

$$\mathcal{P} = \frac{1}{2}\frac{\mathrm{tr}_2\left[(f^+ + f^-)\sigma\right]}{\mathrm{tr}_2\left[f^+ + f^-\right]} = -\frac{1}{2}\tanh(\zeta)\frac{\boldsymbol{P}}{2\zeta}$$

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- **CAUTION**: Valid for small $\omega_{\mu\nu}$ and particle momenta p

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Spin tensor $S^{\lambda,\mu\nu}$

- Total energy-momentum and angular momentum must be fixed

$$P^\mu = \int d^3\Sigma_\lambda \, T^{\lambda\mu} \qquad J^{\mu\nu} = \int d^3\Sigma_\lambda \, J^{\lambda,\mu\nu}$$

- Densities are defined up to divergences
$\Longrightarrow$ **Pseudo-gauge transformations:**

$$T'^{\mu\nu} = T^{\mu\nu} + \frac{1}{2}\partial_\lambda(\Phi^{\lambda,\mu\nu} + \Phi^{\mu,\nu\lambda} + \Phi^{\nu,\mu\lambda})$$

$$S'^{\lambda,\mu\nu} = S^{\lambda,\mu\nu} - \Phi^{\lambda,\mu\nu} + \partial_\alpha Z^{\alpha\lambda,\mu\nu}$$

Leave P^μ and $J^{\mu\nu}$ invariant

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Leave P^{μ} and $J^{\mu\nu}$ invariant

Pauli-Lubański four-vector (I)

- Phase-space density of total angular momentum of particle with momentum p

$$E_p \frac{dJ^{\lambda,\mu\nu}(x,p)}{d^3p}$$

- Pauli-Lubański four-vector

$$E_p \frac{d\Delta\Pi_\mu(x,p)}{d^3p} = -\frac{1}{2}\epsilon_{\mu\nu\alpha\beta}\,\Delta\Sigma_\lambda(x)\,E_p \frac{dJ^{\lambda,\nu\alpha}(x,p)}{d^3p}\,\frac{p^\beta}{m}$$

- Which spin tensor do we use?

$$S^{\lambda,\mu\nu} = \kappa \int \frac{d^3p}{2E_p}\, p^\lambda\, \mathrm{tr}_4\left[(X^+ - X^-)\Sigma^{\mu\nu}\right] = \frac{wu^\lambda}{4\zeta}\omega^{\mu\nu}$$

- Total angular momentum density becomes

$$E_p \frac{dJ^{\lambda,\nu\alpha}(x,p)}{d^3p} = \frac{\kappa}{2}\, p^\lambda\,(x^\nu p^\alpha - x^\alpha p^\nu)\,\mathrm{tr}_4(X^+ + X^-) + \frac{\kappa}{2}\, p^\lambda \mathrm{tr}_4\left[(X^+ - X^-)\,\Sigma^{\nu\alpha}\right]$$

Pauli-Lubański four-vector (I)

- Phase-space density of total angular momentum of particle with momentum p

$$E_p \frac{dJ^{\lambda,\mu\nu}(x,p)}{d^3p}$$

- Pauli-Lubański four-vector

$$E_p \frac{d\Delta\Pi_\mu(x,p)}{d^3p} = -\frac{1}{2}\epsilon_{\mu\nu\alpha\beta}\,\Delta\Sigma_\lambda(x)\,E_p \frac{dJ^{\lambda,\nu\alpha}(x,p)}{d^3p}\,\frac{p^\beta}{m}$$

- **Which spin tensor do we use?**

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- Total angular momentum density becomes

$$E_p \frac{dJ^{\lambda,\nu\alpha}(x,p)}{d^3p} = \frac{\kappa}{2}\,p^\lambda\,(x^\nu p^\alpha - x^\alpha p^\nu)\,\mathrm{tr}_4(X^+ + X^-) + \frac{\kappa}{2}\,p^\lambda \mathrm{tr}_4\left[\left(X^+ - X^-\right)\Sigma^{\nu\alpha}\right]$$

Pauli-Lubański four-vector (II)

- Particle density in the volume $\Delta\Sigma$

$$E_p \frac{d\Delta\mathcal{N}}{d^3p} = \frac{\kappa}{2}\,\Delta\Sigma\cdot p\,\mathrm{tr}_4\left(X^+ + X^-\right)$$

- PL per particle

$$\pi_\mu(x,p) = \frac{\Delta\Pi_\mu(x,p)}{\Delta\mathcal{N}(x,p)}$$

- PL four-vector in the PRF agrees with the Polarization vector (!)

$$\pi_*^0 = 0, \qquad \boldsymbol{\pi}_* = \boldsymbol{\mathcal{P}}$$

- **What about other forms for spin tensor?**

Different form for spin tensor

- PL four-vector

$$E_p \frac{d\Delta\Pi_\mu(x,p)}{d^3p} = -\frac{1}{2}\epsilon_{\mu\nu\alpha\beta}\,\Delta\Sigma_\lambda(x)\,E_p \frac{dJ^{\lambda,\nu\alpha}(x,p)}{d^3p}\frac{p^\beta}{m}$$

- Canonical spin tensor

$$S^{\lambda,\mu\nu}_{\rm can} = \kappa \int \frac{d^3p}{2E_p}\Big(p^\lambda {\rm tr}_4\left[(X^+ - X^-)\Sigma^{\mu\nu}\right] - p^\mu {\rm tr}_4\left[(X^+ - X^-)\Sigma^{\lambda\nu}\right] + p^\nu {\rm tr}_4\left[(X^+ - X^-)\Sigma^{\lambda\mu}\right]\Big)$$

- de Groot, van Leeuwen, van Weert spin tensor

$$S^{\lambda,\mu\nu}_{\rm GLW} = \kappa \int \frac{d^3p}{2E_p} p^\lambda \left({\rm tr}_4\left[(X^+ - X^-)\Sigma^{\mu\nu}\right] + \frac{i}{2m^2}{\rm tr}_4\left[(X^+ - X^-)p_\alpha\gamma^\alpha\left(\gamma^\mu p^\nu - \gamma^\nu p^\mu\right)\right]\right)$$

PL is identical for all forms of spin tensor considered!

Different form for spin tensor

- PL four-vector

$$E_p \frac{d\Delta\Pi_\mu(x,p)}{d^3p} = -\frac{1}{2}\epsilon_{\mu\nu\alpha\beta}\,\Delta\Sigma_\lambda(x)\,E_p \frac{dJ^{\lambda,\nu\alpha}(x,p)}{d^3p}\,\frac{p^\beta}{m}$$

- Canonical spin tensor

$$S^{\lambda,\mu\nu}_{\rm can} = \kappa \int \frac{d^3p}{2E_p}\left(p^\lambda \mathrm{tr}_4\left[(X^+ - X^-)\Sigma^{\mu\nu}\right]\right. \\ \left. - p^\mu \mathrm{tr}_4\left[(X^+ - X^-)\Sigma^{\lambda\nu}\right] + p^\nu \mathrm{tr}_4\left[(X^+ - X^-)\Sigma^{\lambda\mu}\right]\right)$$

- de Groot, van Leeuwen, van Weert spin tensor

$$S^{\lambda,\mu\nu}_{\rm GLW} = \kappa \int \frac{d^3p}{2E_p} p^\lambda \left(\mathrm{tr}_4\left[(X^+ - X^-)\Sigma^{\mu\nu}\right]\right. \\ \left. + \frac{i}{2m^2}\mathrm{tr}_4\left[(X^+ - X^-)p_\alpha\gamma^\alpha\left(\gamma^\mu p^\nu - \gamma^\nu p^\mu\right)\right]\right)$$

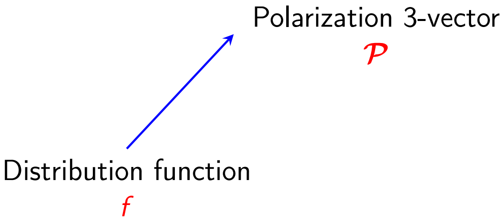
PL is identical for all forms of spin tensor considered!

Objects related to polarization

Distribution function

f

Objects related to polarization



Objects related to polarization

Polarization 3-vector

$\mathcal{P}$

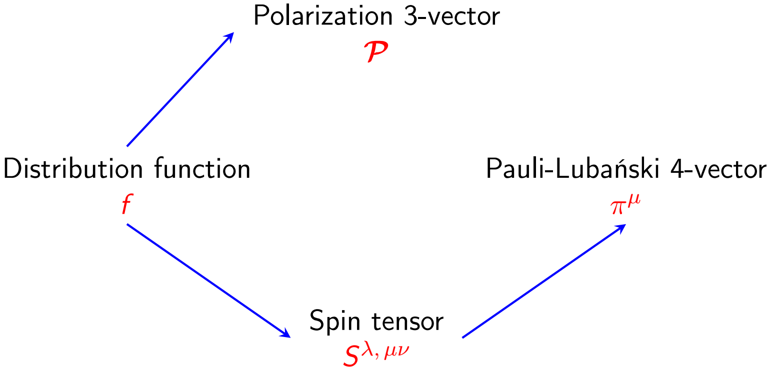
Distribution function

f

Spin tensor

$S^{\lambda,\mu\nu}$

Objects related to polarization



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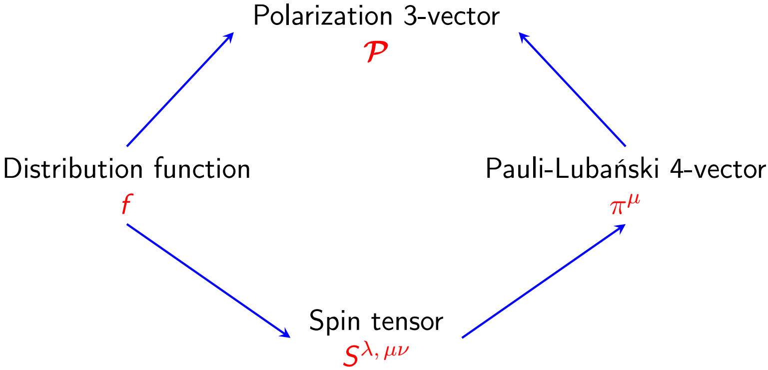

Conclusions

Summary

- Hydrodynamic framework which includes evolution of spin density in a consistent fashion
- Minimal extension of well-established perfect-fluid picture
- Polarization evolution in heavy-ion collisions
- Advantage to study dynamics of systems in **local equilibrium**, compared to studies where global equilibrium was assumed

Outlook

- Spin-orbit interaction
- Dissipative effects
- ...

BACKUP

Global equilibrium and thermal vorticity (I)

Density operator for any quantum mechanical system

$$\rho(t) = \exp\left[-\int d^3\Sigma_\mu(x)\left(T^{\mu\nu}(x)b_\nu(x) - \frac{1}{2}J^{\mu,\alpha\beta}(x)\omega_{\alpha\beta}(x)\right)\right]$$

$d^3\Sigma_\mu$ – element of 3-dimensional hypersurface
Global equilibrium $\Rightarrow$ $\rho(t)$ independent of time

$$\partial_\mu\left(T^{\mu\nu}(x)b_\nu(x) - \frac{1}{2}J^{\mu,\alpha\beta}(x)\omega_{\alpha\beta}(x)\right) = T^{\mu\nu}(x)\left(\partial_\mu b_\nu(x)\right) - \frac{1}{2}J^{\mu,\alpha\beta}(x)\left(\partial_\mu\omega_{\alpha\beta}(x)\right)$$

$$b_\nu = \text{const}\ , \quad \omega_{\alpha\beta} = \text{const}$$

splitting angular momentum into its orbital and spin part

$$\begin{aligned}
\rho &= \exp\left[-\int d^3\Sigma_\mu(x)\left(T^{\mu\nu}(x)b_\nu - \frac{1}{2}\left(L^{\mu,\alpha\beta}(x) + S^{\mu,\alpha\beta}(x)\right)\omega_{\alpha\beta}\right)\right] \\
&= \exp\left[-\int d^3\Sigma_\mu(x)\left(T^{\mu\nu}(x)b_\nu - \frac{1}{2}\left(x^\alpha T^{\mu\beta}(x) - x^\beta T^{\mu\alpha} + S^{\mu,\alpha\beta}(x)\right)\omega_{\alpha\beta}\right)\right] \\
&= \exp\left[-\int d^3\Sigma_\mu(x)\left(T^{\mu\nu}(x)\left(b_\nu + \omega_{\nu\alpha}x^\alpha\right) - \frac{1}{2}S^{\mu,\alpha\beta}(x)\omega_{\alpha\beta}\right)\right]
\end{aligned}$$

Global equilibrium and thermal vorticity (II)

Introducing the notation

$$\beta_\nu = b_\nu + \omega_{\nu\alpha} x^\alpha$$

we may write

$$\rho = \exp\left[-\int d^3\Sigma_\mu(x)\left(T^{\mu\nu}(x)\beta_\nu - \frac{1}{2}S^{\mu,\alpha\beta}(x)\omega_{\alpha\beta}\right)\right]$$

We note that β_ν is the Killing vector, satisfies the equations

$$\partial_\mu\beta_\nu + \partial_\nu\beta_\mu = 0$$
$$-\frac{1}{2}\left(\partial_\mu\beta_\nu - \partial_\nu\beta_\mu\right) = \omega_{\mu\nu} = \text{const}$$