# Kinetic approach to a relativistic Bose-Einstein condensate for massless and massive particles

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# Outline

# Non-equilibrium

- Non-equilibrium thermodynamics is a branch of thermodynamics that deals with physical systems that are not in thermodynamic equilibrium. [Wikipedia]
- Equilibrium thermodynamics ignores the time-courses of physical processes, in contrast non-equilibrium thermodynamics attempts to describe their time-courses in continuous detail. [Wikipedia]
- Thermalisation is the process of physical systems reaching thermal equilibrium through mutual interaction. [Wikipedia]

# BUU-equation

Relativistic evolution equation of a bosonic (including quantum statistics by Bose enhancement) system in non-equilibrium

$$\frac{1}{E_1}\left(p_1^\mu \frac{\partial}{\partial x^\mu} + m \frac{\partial}{\partial p_1^\mu} K_1^\mu\right) f_1 = \frac{1}{2E_1} \int \frac{\mathrm{d}^3 \vec{p_2}}{2(2\pi)^3 E_2} \frac{\mathrm{d}^3 \vec{p_3}}{2(2\pi)^3 E_3} \frac{\mathrm{d}^3 \vec{p_4}}{2(2\pi)^3 E_4} W_{12\leftrightarrow 34} \times \left\{ f_3 f_4 (1+f_1)(1+f_2) - f_1 f_2 (1+f_3)(1+f_4) \right\} =: \mathcal{C}[2 \leftrightarrow 2].$$

$$W_{12\leftrightarrow 34} := (2\pi)^4 \frac{|M_{12\leftrightarrow 34}|^2}{\nu} \delta^{(4)}(P_1 + P_2 - P_3 - P_4).$$

Equilibrium $f_{\mathrm{eq}}$? Detailed balance!

$$\mathcal{C}[2 \leftrightarrow 2] \stackrel{!}{=} 0.$$

# Assumptions

- isotropic system $f(t, \vec{r}, \vec{p}) \rightarrow f(t, \vec{r}, p)$
- homogeneous system $f(t, \vec{r}, p) \rightarrow f(t, p)$
- vanishing external forces $K^\mu = 0$

Detailed balance is satisfied by the Bose-Einstein distribution

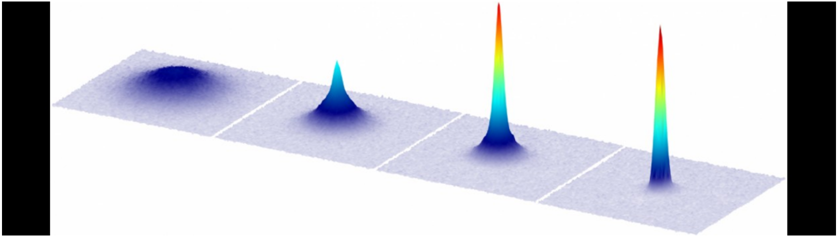
$$f_{\mathrm{eq}}(E_i) = \frac{1}{\exp\left(\frac{E_i-\mu}{T}\right) - 1}.$$

The ground state can become macroscopically large $f_{\mathrm{eq}}(E_0) \gg 1$.
Two cases are considered.

Decreasing the temperature

$$f(E_i) = \frac{1}{\exp\left(\frac{E_i-\mu}{T}\right)-1} \xrightarrow{T\to 0} 0 \quad \text{for} \quad E_i > E_0 \geq \mu.$$

$\longrightarrow$ The occupation number of the ground state $f(E_0)$ becomes macroscopically large



Picture: http://www.erbium.at/FF/wp-content/uploads/2016/01/FirstErbiumBEC-1250x350.jpg

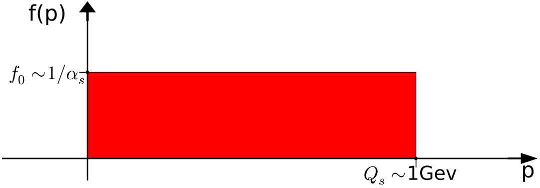
Increasing the particle density

$$f(E_0) = \frac{1}{\exp\left(\frac{E_0-\mu}{T}\right) - 1} \xrightarrow{\mu \to m} \infty \quad \text{for} \quad E_0 = m \geq \mu.$$

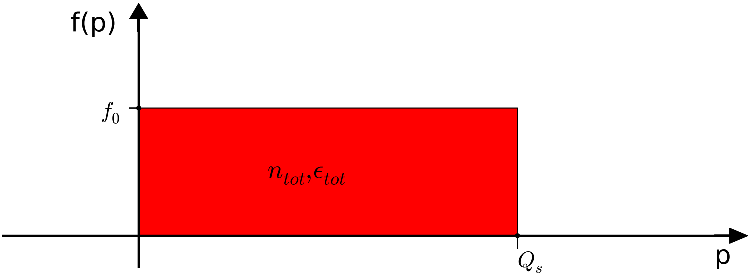
$\longrightarrow$ The occupation number of the ground state $f(E_0)$ becomes macroscopic large

*Can be applied to a very early stage of heavy ion collision:*

$$f(p) \sim 1/\alpha_s \text{ for } p < Q_s$$
$$\simeq f_0 \theta\left(1 - \frac{p}{Q_s}\right)$$



# Determining the equilibrium state




$$n_{\text{tot}} = \int_0^\infty \frac{dp}{2\pi^2} p^2 f_{\text{init}} = \frac{f_0 Q_s^3}{6\pi^2}$$

$$\epsilon_{\text{tot}} = \int_0^\infty \frac{dp}{2\pi^2} p^2 E f_{\text{init}} = \frac{f_0}{16\pi^2} \left\{ Q_s E_{Q_s} (m^2 + 2Q_s^2) + m^4 \log \frac{m}{Q_s + E_{Q_s}} \right\}$$

$$E_{Q_s} := \sqrt{m^2 + Q_s^2}$$

Decompose $f(t, p)$ [Semikoz, Tchakev, arxiv.org/abs/hep-ph/9507306]

$$f(t, \vec{p}) = f_{\text{part}}(t, \vec{p} > 0) + n_c(t)(2\pi)^3\delta^{(3)}(\vec{p})$$

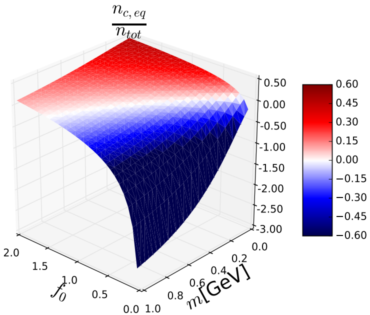
Red known — Blue unknown — Green fixing

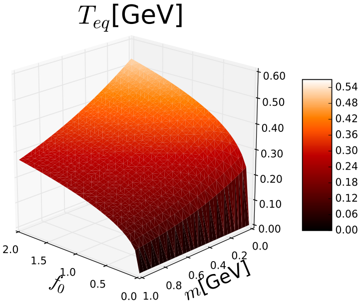
- $n_{\text{tot}} = n_{\text{part,eq}} + n_{\text{c,eq}}$ particle (density) conservation
- $\epsilon_{\text{tot}} = \epsilon_{\text{part,eq}} + \epsilon_{\text{c,eq}}$ energy (density) conservation
- $n_{\text{part,eq}} = \int_0^\infty \frac{\mathrm{d}p}{2\pi^2} p^2 \frac{1}{\exp\left(\frac{E-\mu_{\text{eq}}}{T_{\text{eq}}}\right)-1}$
- $\epsilon_{\text{part,eq}} = \int_0^\infty \frac{\mathrm{d}p}{2\pi^2} p^2 \frac{E}{\exp\left(\frac{E-\mu_{\text{eq}}}{T_{\text{eq}}}\right)-1}$
- $\mu_{\text{eq}} = m$ and $\epsilon_c = n_c m$

Solve for $T_{\text{eq}}$ and $n_{\text{c,eq}}$

# Numerical solution for the equilibrium values

- $Q_s = 1\text{GeV}$




- The blue shaded area suggest a negative condensate density which is not physical
- Condition $\mu_{\text{eq}} = m$ does not apply (underpopulated case)

# massless case $m = 0$

Equilibrium state is given analytically

$$\mu_{\text{eq}} = 0, \qquad \epsilon_{\text{c,eq}} = 0$$

$$\epsilon_{\text{tot,eq}} = \frac{f_0 Q_s^4}{8\pi^2} \overset{!}{=} \frac{\pi^2 T_{\text{eq}}^4}{30} = \epsilon_{\text{part,eq}} \longrightarrow T_{\text{eq}} = \sqrt[4]{f_0 15}\frac{Q_s}{2\pi}$$

$$n_{\text{tot,eq}} = \frac{f_0 Q_s^3}{6\pi^2}, \qquad n_{\text{part,eq}} = (15 f_0)^{\frac{3}{4}} \frac{Q_s^3 \zeta(3)}{2\sqrt{2}\pi^5}$$

$$n_{\text{c,eq}} = \frac{f_0 Q_s^3}{6\pi^2} - (15 f_0)^{\frac{3}{4}} \frac{Q_s^3 \zeta(3)}{2\sqrt{2}\pi^5}$$

$$n_{\text{c,eq}} = 0 \longrightarrow f_0 \approx 0.154 \text{ (the critical case)}$$

# Equilibrium for underpopulated systems

Solve for $T_{\text{eq}}$ and $\mu_{\text{eq}}$:

$$n_{\text{tot}} = n_{\text{part,eq}} = \int_0^\infty \frac{dp}{2\pi^2} p^2 f_{\text{eq}}(\mu_{\text{eq}}, T_{\text{eq}})$$
$$\epsilon_{\text{tot}} = \epsilon_{\text{part,eq}} = \int_0^\infty \frac{dp}{2\pi^2} p^2 E f_{\text{eq}}(\mu_{\text{eq}}, T_{\text{eq}})$$

$$f(t,\vec{p}) \to f(t,\vec{p}) = f_{\text{part}}(t,\vec{p}>0) + \underbrace{n_c(t)(2\pi)^3\delta^{(3)}(\vec{p})}_{=:f_c}$$

- Set of 2 coupled first order differential equation.

Evolution equation for the BEC

$$\frac{\partial f_{c,1}}{\partial t} = \frac{1}{2E_1}\int \frac{\mathrm{d}^3\vec{p_2}}{2(2\pi)^3E_2}\frac{\mathrm{d}^3\vec{p_3}}{2(2\pi)^3E_3}\frac{\mathrm{d}^3\vec{p_4}}{2(2\pi)^3E_4} \times(2\pi)^4\frac{|M_{12\leftrightarrow 34}|^2}{\nu}\delta^4(P_1+P_2-P_3-P_4) \times\{f_{c,1}f_3f_4 - f_{c,1}f_2(1+f_3+f_4)\} =: \mathcal{C}[1c+1\leftrightarrow 2]$$

Following a integration over $\vec{p}_1$

$$\frac{\partial n_c}{\partial t} = \int \frac{\mathrm{d}^3\vec{p}_1}{(2\pi)^3}\mathcal{C}[1c+1\leftrightarrow 2]$$

## Evolution equation for the higher modes

$$
\begin{aligned}
\frac{\partial f_1}{\partial t} = \frac{1}{2E_1} \int & \frac{\mathrm{d}^3\vec{p_2}}{2(2\pi)^3 E_2} \frac{\mathrm{d}^3\vec{p_3}}{2(2\pi)^3 E_3} \frac{\mathrm{d}^3\vec{p_4}}{2(2\pi)^3 E_4} \\
& \times (2\pi)^4 \frac{|M_{12\leftrightarrow 34}|^2}{\nu} \delta^4(P_1 + P_2 - P_3 - P_4) \\
& \times \Big[ \{ f_3 f_4 (f_1 + 1)(f_2 + 1) - f_1 f_2 (f_3 + 1)(f_4 + 1) \} \\
& + \{ f_{c,2} f_3 f_4 - f_{c,2} f_1 (1 + f_3 + f_4) \} \\
& + \{ (1 + f_1 + f_2) f_{c,3} f_4 - f_{c,3} f_1 f_2 \} \\
& + \{ (1 + f_1 + f_2) f_{c,4} f_3 - f_{c,4} f_1 f_2 \} \Big] \\
:= \mathcal{C}[2 \leftrightarrow 2] + \mathcal{C}[1 + 1\mathrm{c} \leftrightarrow 2]
\end{aligned}
$$

- 9 dimensional Integrals are not practical for our numerical approach
- $\frac{|M_{12\leftrightarrow 34}|^2}{\nu} \propto s = (P_1 + P_2)^2 \rightarrow$ integrate out every angular dependencies and also the internal momenta $\tilde{p}_4$

$$\frac{\partial n_c}{\partial t} = n_c \frac{9\lambda^2}{64\pi^3} \int_0^\infty \mathrm{d}p_2 \int_0^\infty \mathrm{d}p_3 \frac{p_2 p_3}{m_1 E_2 E_3} \times \left[-1 - \epsilon(p_2 - p_3 - \tilde{p}_4) + \epsilon(p_2 + p_3 - \tilde{p}_4) + \epsilon(p_2 - p_3 + \tilde{p}_4)\right] \times \left(m_1^2 + m_2^2 + 2m_1 E_2\right) \theta(\tilde{p}_4^2)[f_3 f_4 - f_2(1 + f_3 + f_4)]$$

$$\frac{\partial f_1}{\partial t} = \underbrace{\mathcal{C}[2 \leftrightarrow 2]}_{\int_{\mathbb{R}^+ \times \mathbb{R}^+}} + \underbrace{\mathcal{C}[1 + 1c \leftrightarrow 2]}_{3\times \int_{\mathbb{R}^+},\ \propto n_c}$$

# What we have so far

A given initial state with the mass of the particles m:

A final determined state:

$$f_{\text{init}}(p) = f_0 \theta\left(1 - \frac{p}{Q_s}\right) \xrightarrow{EoM} f_{\text{eq}}(p) = \frac{1}{\exp\left(\frac{\sqrt{p^2+m^2}-\mu_{\text{eq}}}{T_{\text{eq}}}\right) - 1} + n_{c,\text{eq}}(2\pi)^3\delta^{(3)}(\vec{p})$$

- 2 first order coupled integro-differential equations
- $\rightarrow \frac{|M_{12\leftrightarrow 34}|^2}{\nu} \propto s = (P_1 + P_2)^2$ corresponds to $\sigma_{\text{tot}} = \text{const.}$
- Inclusion of massles condensate particles is possible in contrast to $\frac{|M_{12\leftrightarrow 34}|^2}{\nu} \propto const.$ [arXiv:1510.04552]
- Analytic solution? Researched field - [arxiv:1507.07834]
- Numerical evaluation!

$$f_{i+1}^{\text{Euler}} = f_i^{\text{Euler}} + k_i$$

$$f_{i+1}^{\text{RK4}} = f_i^{\text{RK4}} + \frac{2825}{27648}k_{i,1} + \frac{18575}{48384}k_{i,3} + \frac{13525}{55296}k_{i,4} - \frac{277}{14336}k_{i,5} + \frac{1}{4}k_{i,6}$$
$$f_{i+1}^{\text{RK5}} = f_i^{\text{RK5}} + \frac{37}{378}k_{i,1} + \frac{250}{621}k_{i,3} + \frac{125}{594}k_{i,4} - \frac{1}{5}k_{i,6}$$

[Transactions on Mathematical Software 16: 201-222, 1990. doi:10.1145/79505.79507]

- two approximations of order 4 and 5
- no additional computation time for the second approximation
- compare the approximations
- Cash-Karp method involves $h_{\text{new}} = sh_{\text{old}}$

$$s = \left| \frac{\epsilon_{\text{tol}}}{f_{i+1}^{\text{RK5}} - f_{i+1}^{\text{RK4}}} \right|^{\frac{1}{5}}$$

# Applying on the Boltzmann-equation

- $f_{\text{part}}(p)$ is given on a Grid $G := \{p^0, p^1, ..., p^i, ..., p^N\}$ with $p^0 < p^1 < ... < p^i < ... < p^N$ $(N > 100)$
- then solve the Boltzmann equation independently for every Grid point (external momenta $p^i$) by applying the RKCK45 scheme.
- to evaluate the collision integrals we apply quadrature methods (trapezoidal, Simpson)

# Condensation onset

onset:= Starting time of condensation

- the condensation process $\dot{n}_c \propto n_c$ happens only for $n_c \neq 0$
- BEC is a phenomena which arises due to fluctuations and are not included in this approach
- two Possibilities to include condensation are:

  1. initialising with a finite but negligibly small condensate seed $n_c \ll n_{\mathrm{tot}}$
  2. inserting a small condensate seed $n_c \ll n_{\mathrm{tot}}$ when the distribution function reaches a certain point

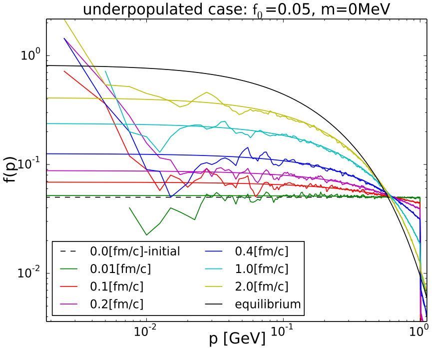
- extraction of 2 parameters ($\mu_{\mathrm{eff}}, T_{\mathrm{eff}}$) by fitting the Bose distribution to $f_{\mathrm{part}}$ and then inserting the seed when the condition $\mu_{\mathrm{eff}} = m$ is given

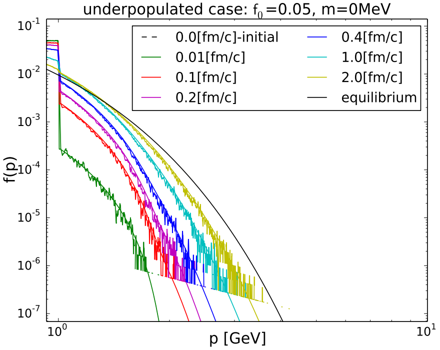
# Before The result:

- Transport code BAMPS [Greiner arXiv:hep-ph/0406278]
  (= Boltzmann Approach to Multi-Parton Scatterings)
- Test particle Ansatz
- Box calculatation

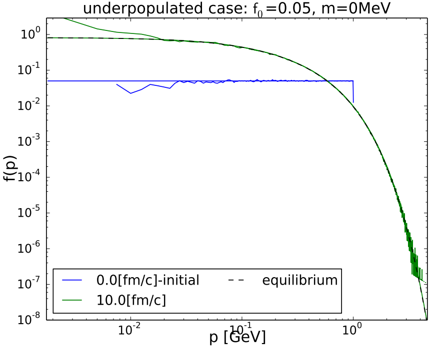
Our code: smooth lines
BAMPS: shaky lines

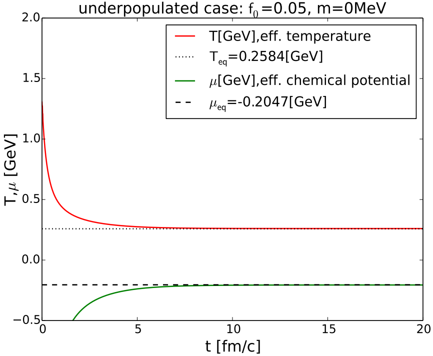
# Underpopulated massless case



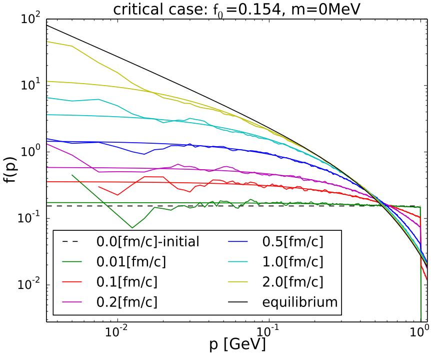
# Underpopulated massless case

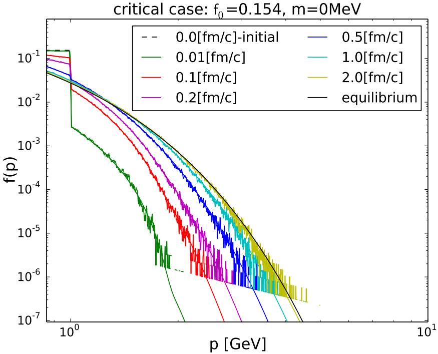


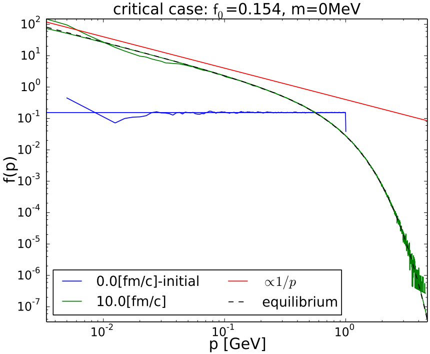

- No condensation since $\mu_{\text{eff}} < m$

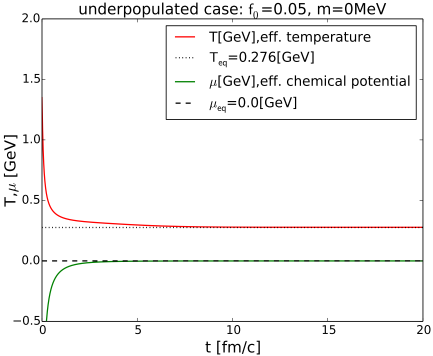
# Critical massless case



# Critical massless case

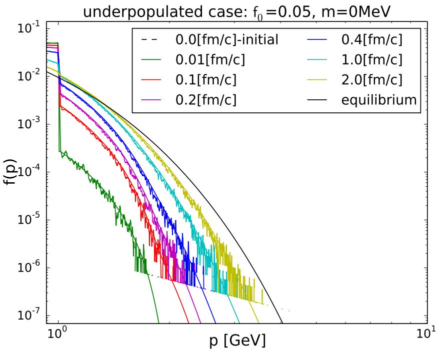


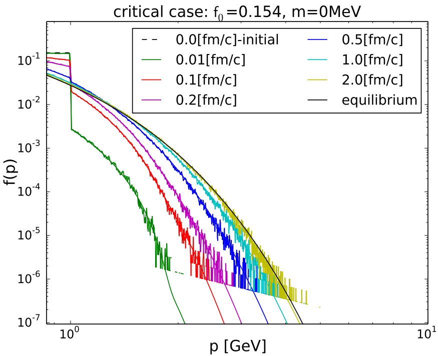




- No condensation since $\mu_{\text{eff}}$ converges to the mass (0GeV)

# Clip

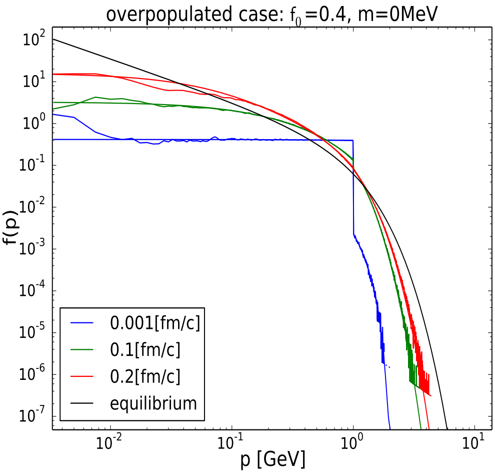
# Thermalisation time




- focus $f(t = 2.0[\mathrm{fm/c}], p)$ XXXXX
- Increasing the total particle density leads to a faster thermalisation-consistent

# Overpopulated massless case



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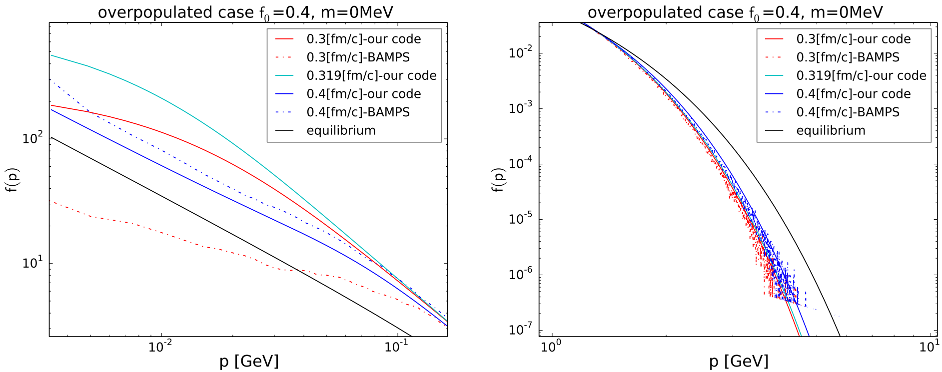
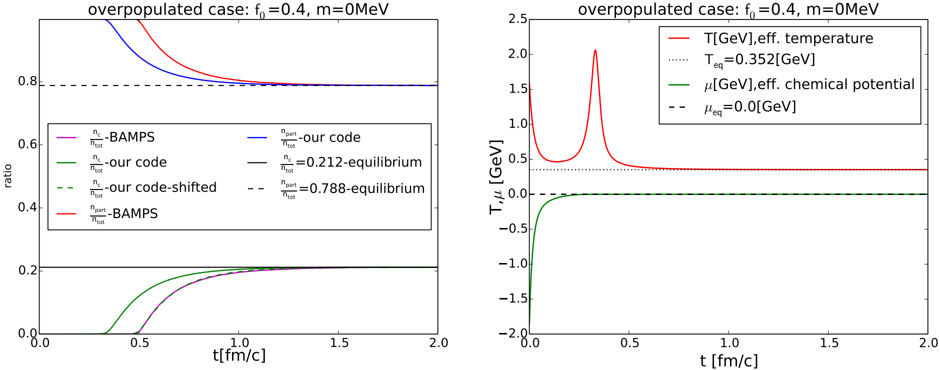
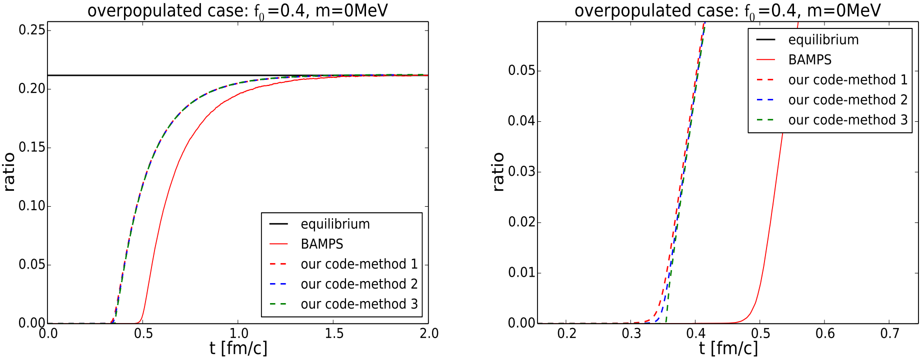



- Maybe just a minor bug? A time-shift?

# Condensate evolution



- About the onset (onset:= Starting time of condensation)

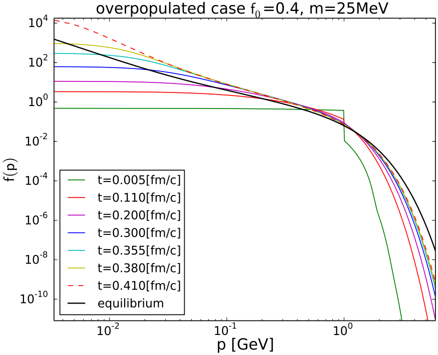



- method 1: Starting with condensate seed
- method 2: Inserting a seed when $\mu_{\text{eff}} = 0$
- method 3: Any time

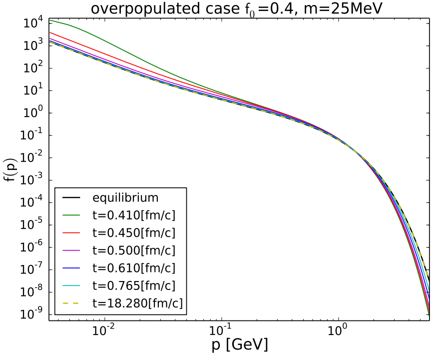
# Clip

# Overpopulated massive $m = 25MeV$ case

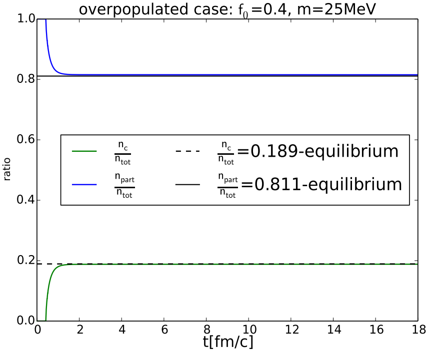
## Before the onset

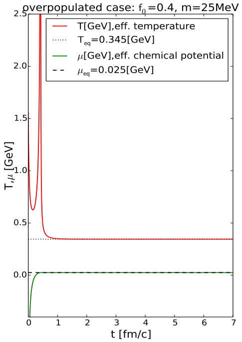



## After the onset



# Overpopulated massive $m = 25MeV$ case



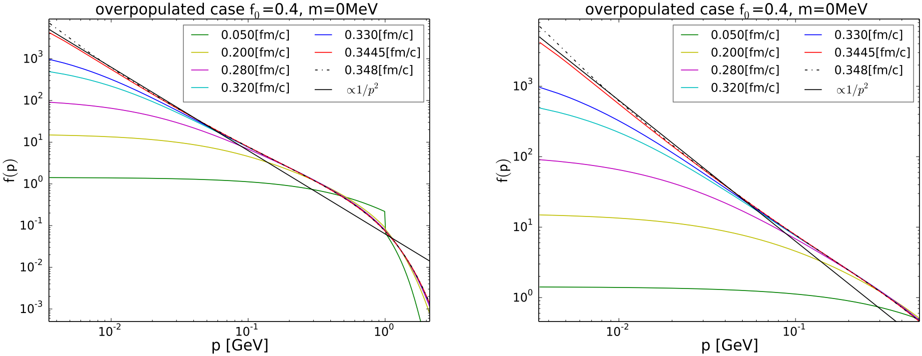



# Conclusion and Outlook

- All our simulations thermalize into equilibrium.
- The evolution of our system without condensate $\dot{f}_1 = \mathcal{C}[2 \leftrightarrow 2]$ is in a good agreement with BAMPS.
- Overpopulated systems differs by a time shift later while Equilibration is still given
- The different onset methods are equivalent.

- A new BAMPS run is going on to set the onset manually at a earlier time.
- Further numerical tests have to include a comparison with the analytic solution for a classical system in [arxiv:1507.07834]:
- Adapting this scheme for longitudinal expanding (anisotropic) systems (Bjorken coordinates).

Thank You!


overpopulated case $f_0$=0.4, m=0MeV
f(p)
p [GeV]
0.050[fm/c]
0.200[fm/c]
0.280[fm/c]
0.320[fm/c]
0.330[fm/c]
0.3445[fm/c]
0.348[fm/c]
$\propto 1/p^2$
overpopulated case $f_0$=0.4, m=0MeV
f(p)
p [GeV]
0.050[fm/c]
0.200[fm/c]
0.280[fm/c]
0.320[fm/c]
0.330[fm/c]
0.3445[fm/c]
0.348[fm/c]
$\propto 1/p^2$

# Eulers method

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \mathcal{C}(t, f) \quad , \quad f(t_0) = f_0$$

$$\left.\frac{\mathrm{d}f}{\mathrm{d}t}\right|_{t=t_i} = \mathcal{C}(t_i, f_i)$$

$$f_t = f_0 + \mathcal{C}(t_0, f_0)(t - t_0)$$

$$f_{i+1} = f_i + \mathcal{C}(t_i, f_i)(t_{i+1} - t_i)$$

Using a uniform step size $h := t_{i+1} + t_i = const.$ and substituting $k_i = h\mathcal{C}(t_i, f_i)$ one ends up ith:

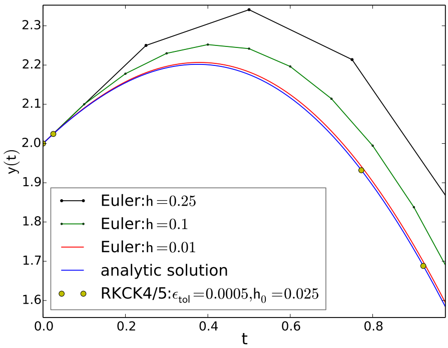
$$f_{i+1}^{\mathrm{Euler}} = f_i + k_i$$

# Cash-Karp RK45 -scheme

starting point: 2 approximations whereby both approximations need the evaluation of the following six values

$$
\begin{aligned}
k_{i,1} &= h\mathcal{C}(t_i, f_i) \\
k_{i,2} &= h\mathcal{C}\left(t_i + \frac{1}{5}h, f_i + \frac{1}{5}k_{i,1}\right) \\
k_{i,3} &= h\mathcal{C}\left(t_i + \frac{3}{10}h, f_i + \frac{3}{40}k_{i,1} + \frac{9}{40}k_{i,2}\right) \\
k_{i,4} &= h\mathcal{C}\left(t_i + \frac{3}{5}h, f_i + \frac{3}{10}k_{i,1} - \frac{9}{10}k_{i,2} + \frac{6}{5}k_{i,3}\right) \\
k_{i,5} &= h\mathcal{C}\left(t_i + h, f_i - \frac{11}{54}k_{i,1} + \frac{5}{2}k_{i,2} - \frac{70}{27}k_{i,3} + \frac{35}{27}k_{i,4}\right) \\
k_{i,6} &= h\mathcal{C}\left(t_i + \frac{7}{8}h, f_i + \frac{1631}{55296}k_{i,1} + \frac{175}{512}k_{i,2} + \frac{575}{13824}k_{i,3} + \frac{44275}{110592}k_{i,4} - \frac{253}{4096}k_{i,5}\right)
\end{aligned}
$$

# illustration




Given an first order ODE with the following IVP

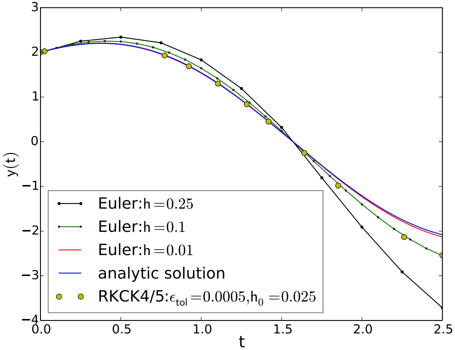
$$\frac{df}{dt} = \cos^2(t) + \tan(t)f$$

$$f(0) = 2$$

and its analytic solution

$$f(t) = [\sin(t)+2]\cos(t)$$

# illustration




Given an first order ODE with the following IVP

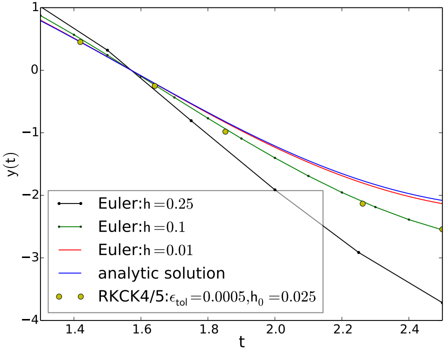
$$\frac{\mathrm{d}f}{\mathrm{d}t} = \cos^2(t) + \tan(t)f$$

$$f(0) = 2$$

and its analytic solution

$$f(t) = [\sin(t)+2]\cos(t)$$

# illustration

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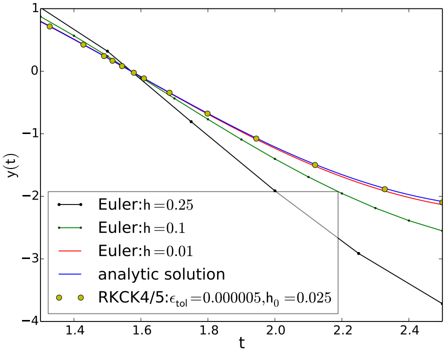
# illustration

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# Overpopulated massless case $f_0 = 0.4$

- our code initialised with a seed and in BAMPS the condensate evolution is prohibited