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Exercise Sheet 11

Problem 1: Color SU(3)

We consider color states for quarks $|c\rangle$ ($c \in \{r, g, b\}$ and anti-quarks $|\bar{c}\rangle$ ($c \in \{\overline{r}, \overline{g}, \overline{b}\}$). They transform under the fundamental 3-representation and its conjugate complex 3^*-representation respectively, i.e., for any $\hat{U} \in \mathrm{SU}(3)$

$$\left|c'\right\rangle = \sum_c |c\rangle\, U_{cc'}, \quad \left|\bar{c}'\right\rangle = \sum_{c=1}^{3} |\bar{c}\rangle\, U^*_{cc'}. \tag{1}$$

Hadrons must be color singlets, i.e., they must be states that do not change under color-SU(3) transformations. Show that...

(a) ...for a meson, i.e., a bound state of a quark and an antiquark, the color state

$$\left|\psi_{c,\text{meson}}\right\rangle = \sum_{c_1,c_2} C_{c_1 c_2} \left|\bar{c}_1, c_2\right\rangle \tag{2}$$

is determined by $C_{c_1 c_2} = \delta_{c_1 c_2}$.

Solution: using (1) the transformation of the meson-color state is

$$\left|\psi'_{c,\text{meson}}\right\rangle = \sum_{c_1,c_2,j,k} C_{c_1 c_2} \left|\bar{j}, \bar{k}\right\rangle U^*_{jc_1} U_{kc_2}. \tag{3}$$

Now with $\hat{C} = (C_{c_1 c_2})$ we can write

$$\sum_{c_1,c_2} U^*_{jc_1} C_{c_1 c_2} U_{kc_2} = \sum_{c_1,c_2} (\hat{U}^\dagger)^*_{jc_1} C_{c_1 c_2} U_{kc_2} = (\hat{U}^\dagger \hat{C} \hat{U})_{jk} \stackrel{!}{=} \delta_{jk}. \tag{4}$$

This means that

$$\hat{U}^\dagger \hat{C} = \hat{U}^{-1} = \hat{U}^\dagger \tag{5}$$

multiplying this by $\hat{U}$ from the left gives

$$\hat{C} = \mathbb{1} \Rightarrow C_{jk} = \delta_{jk}. \tag{6}$$

(b) ...for a baryon, i.e., a bound state of three quarks, the color state is the totally antisymmetric state

$$\left|\psi_{c,\text{baryon}}\right\rangle = \sum_{c_1,c_2,c_3} \epsilon_{c_1 c_2 c_3} \left|c_1, c_2, c_2\right\rangle, \tag{7}$$

where ϵ_{jkl} is the usual totally antisymmetric Levi-Civita symbol with $\epsilon_{rgb} = 1$.

Solution: The given baryon-color state transforms as

$$\left|\psi'_{c,\text{baryon}}\right\rangle = \sum_{c_1,c_2,c_3} \sum_{j,k,l} \epsilon_{c_1 c_2 c_3} \left|j, k, l\right\rangle U_{jc_1} U_{kc_2} U_{lc_3}. \tag{8}$$

Now

$$\sum_{c_1,c_2,c_3} \epsilon_{c_1 c_2 c_3} \left|j, k, l\right\rangle U_{jc_1} U_{kc_2} U_{lc_3} = \det \hat{U} \epsilon_{jkl}, \tag{9}$$

i.e. $\left|\psi'_{c,\text{baryon}}\right\rangle = \left|\psi_{c,\text{baryon}}\right\rangle$, as it must be.

Note: The representation theory of SU(3) shows that this is the only color-neutral combination for three-quark states.

(c) Consider the three antisymmetrized "di-quark"color state

$$\left|\psi_{c_1}\right\rangle = \sum_{c_2,c_3} \epsilon_{c_1 c_2 c_3} \left|c_2, c_3\right\rangle. \tag{10}$$

Show that it transforms under the conjugate-complex 3^* representation of the color SU(3).

Hint: Note that $\det \hat{U} = 1$, which implies that

$$\sum_{j,k,l} \epsilon_{jkl} U_{jj'} U_{kk'} U_{ll'} = \sum_{j,k,l} \epsilon_{jkl} U_{j'j} U_{k'k} U_{l'l} = \det \hat{U} \epsilon_{j'k'l'}. \tag{11}$$

Solution: The di-quark color states transform as

$$\left|\psi'_{c_1}\right\rangle = \sum_{c_2,c_3} \sum_{j,k} \epsilon_{c_1 c_2 c_3} \left|j,k\right\rangle U_{jc_2} U_{kc_3}. \tag{12}$$

Now we

$$\tilde{U}_{c_1 jk} = \sum_{c_2,c_3} \epsilon_{c_1 c_2 c_3} U_{jc_2} U_{kc_3}. \tag{13}$$

From this we have

$$\sum_{c_1} U_{ic_1} \tilde{U}_{c_1 jk} = (\hat{U} \hat{\tilde{U}}_k)_{ij} = \sum_{c_1,c_2,c_3} \epsilon_{c_1 c_2 c_3} U_{ic_1} U_{jc_2} U_{kc_3} = \det \hat{U} \epsilon_{ijk} = \epsilon_{ijk}. \tag{14}$$

Thus, because of $\hat{U}^\dagger = \hat{U}^{-1}$

$$\tilde{U}_{c_1 jk} = \sum_i (\hat{U}^\dagger)_{c_1 i} (\hat{U} \hat{\tilde{U}}_k)_{ij} = \sum_i U^*_{ic_1} \epsilon_{ijk} \tag{15}$$

and finally, using this in (12) applying (13)

$$\left|\psi'_{c_1}\right\rangle = \sum_{j,k} \tilde{U}_{c_1 jk} \left|j,k\right\rangle = \sum_{i,j,k} \epsilon_{ijk} \left|j,k\right\rangle U^*_{ic_1} = \sum_i \left|\psi_i\right\rangle U^*_{ic_1}, \tag{16}$$

i.e., according to (1) the antisymmetrized di-quark state indeed transforms unter the 3^*-representation of SU(3).

Problem 2: Some isospin gymnastics

In this exercise we want to investigate the representation theory of SU(2) (in the physical interpreataion as "isospin"), using Schwinger's harmonic-oscillator approach, corresponding to a two-flavor-quark model (describing the two lightest u- and d-Quarks).

A symmetric harmonic oscillator in 2 dimensions can be described by the Hamiltonian

$$\mathbf{H} = \omega \mathbf{N}, \quad \mathbf{N} = \mathbf{N}_1 + \mathbf{N}_2, \quad \mathbf{N}_f = \mathbf{a}_f^\dagger \mathbf{a}_f, \quad f \in \{1,2\}. \tag{17}$$

where the $\mathbf{a}_f$ obey the commutation relations of bosonic annihilation operators of a field mode, which is the connection with QFT:

$$\left[\mathbf{a}_f, \mathbf{a}_g\right] = 0, \quad \left[\mathbf{a}_f, \mathbf{a}_g^\dagger\right] = \delta_{fg} \mathbb{1}. \tag{18}$$

For the energy-eigenvalue problem we can choose the flavor-number operators $\mathbf{N}_f$ as a complete set of compatible observables, and from the quantum mechanics of harmonic oscillators we know that the corresponding complete orthonormal set of energy eigenstates can be constructed from the "vacuum" $|\Omega\rangle = |N_1 = 0, N_2 = 0\rangle$, defined by

$$\mathbf{a}_f |\Omega\rangle = 0 \tag{19}$$

for all f and

$$|N_1, N_2\rangle = \prod_{j=1}^{2} \frac{1}{\sqrt{N_j!}} \left(\mathbf{a}_f^\dagger\right)^{N_j} |\Omega\rangle. \tag{20}$$

It is now clear that the harmonic oscillator has SU(2) as a symmetry group, i.e., the Hamiltonian is invariant under the transformation

$$\mathbf{a}'_f = \sum_{g=1}^{2} U_{fg} \mathbf{a}_g \tag{21}$$

with $\hat{U} \in \mathrm{SU}(2)$, which implies that

$$\mathbf{a}_f'^\dagger = \sum_{g=1}^{2} \mathbf{a}_g^\dagger U_{fg}^* = \sum_{g=1}^{2} \mathbf{a}_g^\dagger (\hat{U}^\dagger)_{gf}. \tag{22}$$

We want to show that the familiar irreducible spin represenations with $s \in \{0, 1/2, \ldots\}$ can be constructed by the action of creation and annihilation operators on the complete set of orthonormal eigenfunctions $|N_1, N_2\rangle$ with fixed total number $N = N_1 + N_2$. Obviously the three operators

$$\mathbf{s}_+ = \mathbf{a}_1^\dagger \mathbf{a}_2, \quad \mathbf{s}_- = \mathbf{a}_2^\dagger \mathbf{a}_1 = \mathbf{s}_-^\dagger, \quad \mathbf{s}_3 = \frac{1}{2}(\mathbf{N}_1 - \mathbf{N}_2) \tag{23}$$

leave the eigenspaces of $\mathbf{N}$ with eigenvalues $N \in \{0, 1, 2, \ldots\}$ invariant.
Show that the three operators

$$\mathbf{s}_1 = \frac{1}{2}(\mathbf{s}_+ + \mathbf{s}_-), \quad \mathbf{s}_2 = \frac{1}{2\mathrm{i}}(\mathbf{s}_+ - \mathbf{s}_-), \quad \mathbf{s}_3 \tag{24}$$

fulfill the commutation relations of angular-momentum components

$$\left[\mathbf{s}_j, \mathbf{s}_k\right] = \mathrm{i}\epsilon_{jkl} \mathbf{s}_l. \tag{25}$$

Calculate $\vec{\mathbf{s}}^2$ and show that the operation of this isospin algebra on the eigenspace of $\mathbf{N}$ with eigenvalue N realize the known irreducible representations of the su(2) algebra with $s = N/2$, where $s(s+1)$ are the eigenvalues of $\vec{\mathbf{s}}^2$.
Solution: We start by calculating the commutators of $\mathbf{s}_\pm$ and $\mathbf{s}_3$:

$$\left[\mathbf{s}_+, \mathbf{s}_+\right] = \left[\mathbf{a}_1^\dagger \mathbf{a}_2, \mathbf{a}_1^\dagger \mathbf{a}_2\right] = 0, \tag{26}$$

because $\mathbf{a}_1$ and $\mathbf{a}_2^\dagger$ commute. For the same reason we also have

$$\left[\mathbf{s}_-, \mathbf{s}_-\right] = 0. \tag{27}$$

Further, using the general formulae

$$[\mathbf{AB}, \mathbf{C}] = \mathbf{A}[\mathbf{B}, \mathbf{C}] + [\mathbf{A}, \mathbf{C}]\mathbf{B}, \quad [\mathbf{A}, \mathbf{BC}] = \mathbf{B}[\mathbf{A}, \mathbf{C}] + [\mathbf{A}, \mathbf{B}]\mathbf{C}. \tag{28}$$

$$
\begin{aligned}
\left[\mathbf{s}_+,\mathbf{s}_-\right] &= \left[\mathbf{a}_1^\dagger\mathbf{a}_2,\mathbf{a}_1\mathbf{a}_2^\dagger\right] = \mathbf{a}_1^\dagger\left[\mathbf{a}_2,\mathbf{a}_1\mathbf{a}_2^\dagger\right] + \left[\mathbf{a}_1^\dagger,\mathbf{a}_1\mathbf{a}_2^\dagger\right]\mathbf{a}_2 = \mathbf{a}_1^\dagger\mathbf{a}_1\left[\mathbf{a}_2,\mathbf{a}_2^\dagger\right] + \left[\mathbf{a}_1^\dagger,\mathbf{a}_1\right]\mathbf{a}_2^\dagger\mathbf{a}_2 = \mathbf{N}_1 - \mathbf{N}_2 = 2\mathbf{s}_3, && (29)\\
\left[\mathbf{s}_+,\mathbf{s}_3\right] &= \frac{1}{2}\left[\mathbf{a}_1^\dagger\mathbf{a}_2,\mathbf{a}_1{}^\dagger\mathbf{a}_1 - \mathbf{a}_2{}^\dagger\mathbf{a}_2\right] = -\mathbf{s}_+, && (30)\\
\left[\mathbf{s}_-,\mathbf{s}_3\right] &= -\left[\mathbf{s}_3,\mathbf{s}_+\right]^\dagger = +\mathbf{s}_-. && (31)
\end{aligned}
$$

Then with (24) one indeed finds the commutator relations (25), i.e., that $\hat{\vec{s}}$ indeed fulfill the angular-momentum algebra, as it should be for spin components.

Then

$$
\begin{aligned}
\vec{\mathbf{s}}^2 &= \frac{1}{4}(\mathbf{s}_+ + \mathbf{s}_-)^2 - \frac{1}{4}(\mathbf{s}_+ - \mathbf{s}_-)^2 + \mathbf{s}_3^2\\
&= \frac{1}{4}(\mathbf{s}_+^2 + \mathbf{s}_-^2 + \mathbf{s}_+\mathbf{s}_- + \mathbf{s}_-\mathbf{s}_+) - \frac{1}{4}(\mathbf{s}_+^2 + \mathbf{s}_-^2 - \mathbf{s}_+\mathbf{s}_- - \mathbf{s}_-\mathbf{s}_+) + \mathbf{s}_3^2\\
&= \frac{1}{2}(\mathbf{s}_-\mathbf{s}_+ + \mathbf{s}_+\mathbf{s}_-) + \mathbf{s}_3^2.
\end{aligned}
\tag{32}
$$

Now

$$
\begin{aligned}
\mathbf{s}_-\mathbf{s}_+ &= \mathbf{a}_1\mathbf{a}_2^\dagger\mathbf{a}_1^\dagger\mathbf{a}_2 = \mathbf{a}_1\mathbf{a}_1^\dagger\mathbf{N}_2 = (\left[\mathbf{a}_1,\mathbf{a}_1^\dagger\right] + \mathbf{N}_1)\mathbf{N}_2 = (\mathbf{N}_1 + 1)\mathbf{N}_2,\\
\mathbf{s}_+\mathbf{s}_- &= \mathbf{a}_1^\dagger\mathbf{a}_2\mathbf{a}_1\mathbf{a}_2^\dagger = \mathbf{N}_1(\mathbf{N}_2 + 1).
\end{aligned}
\tag{33}
$$

Plugging this into (32) we get

$$
\vec{\mathbf{s}}^2 = \mathbf{N}_1\mathbf{N}_2 + \frac{1}{2}(\mathbf{N}_1 + \mathbf{N}_2) + \frac{1}{4}(\mathbf{N}_1^2 + \mathbf{N}_2^2 - 2\mathbf{N}_1\mathbf{N}_2) = \frac{1}{4}\mathbf{N}^2 + \frac{1}{2}\mathbf{N} = \frac{\mathbf{N}}{2}\left(\frac{\mathbf{N}}{2} + 1\right). \tag{34}
$$

This implies that $|N_1,N_2\rangle$ with $N_1 + N_2$ are simultaneous eigen vectors of $\vec{\mathbf{s}}^2$ with eigenvalue $s(s+1)$ with $s = N/2$ and of $\mathbf{s}_3$ with eigen values $(N_1 - N_2)/2$. Since the maximal possible values of N_1 and N_2 in the subspace with $N_1 + N_2 = N$ are both N the eigenvalues of $\mathbf{s}_3$ at given $s = N/2$ are $m_s \in \{s, s-1, \ldots, -s\}$ and one can start from the eigenvector with $m_s = -s$, i.e., $|0,N\rangle$ ans use $\mathbf{s}_-$ to get all the other eigenvectors of this subspace,

$$
|0,N\rangle, \quad |1,N-1\rangle, \quad |2,N-2\rangle, \quad \ldots, \quad |N,0\rangle. \tag{35}
$$

This implies that these representations of the angular-momentum Lie algebra su(2) are irreducible.

All this is of course also well known from the usual treatment of the angular-momentum eigenvalue problem using only the commutation relations (25) [Hee15].

Remark on the conjugate complex fundamental representation

The anti-particle operators by definition transform with $\hat{U}^*$ with generators $-\hat{t}^*$. Show however that this realizes a representation which is unitarily equivalent to the representation with $s = 1/2$.

Hint: Use the known Pauli matrices and show that $-\hat{\vec{\sigma}}^* = \hat{\sigma}_2\hat{\vec{\sigma}}\hat{\sigma}_2$.

Solution: The Pauli matrices are

$$
\hat{\sigma}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \hat{\sigma}_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \hat{\sigma}_3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{36}
$$

They fulfill the anti-commutator relations

$$
\left\{\hat{\sigma}_j, \hat{\sigma}_k\right\} = 2\delta_{jk}\mathbb{1}. \tag{37}
$$

From that it follows $\boldsymbol{\sigma}_2^2 = \mathbb{1} = \boldsymbol{\sigma}_2\boldsymbol{\sigma}_2^\dagger$, i.e., $\boldsymbol{\sigma}_2$ is both a self-adjoint and unitary matrix, and the equations

$$
\begin{aligned}
-\hat{\sigma}_1^* &= -\hat{\sigma}_1 = -\hat{\sigma}_2^2\hat{\sigma}_1 = +\hat{\sigma}_2\hat{\sigma}_1\hat{\sigma}_2,\\
-\hat{\sigma}_2^* &= \hat{\sigma}_2 = \hat{\sigma}_3^3,\\
-\hat{\sigma}_3^* &= -\hat{\sigma}_3 = -\hat{\sigma}_2^2\hat{\sigma}_3 = \hat{\sigma}_2\hat{\sigma}_3\hat{\sigma}
\end{aligned}
\tag{38}
$$

show indeed the complex-conjugate of the fundamental representation of SU(2) is unitarily equivalent to the fundamental representation.

This is also clear without this explicit calculation since we already know that there is only one 2-dimensional representation of SU(2).

Note: With the same technique one can also treat the representation theory of the SU(N) group for any $N \in \mathbb{N}$. For $N \geq 3$ there are two inequivalent fundamental representations with two distinct kinds of SU(N)-spinors, the one kind transforming with $\hat{U} \in \mathrm{SU}(N)$ and the other with $\hat{U}^*$, i.e., the conjugate complex representation, i.e., for more than 2 flavors the flavor-symmetry operations and the corresponding non-zero hypercharges are different for quarks and antiquarks (for u- and d-quarks all hypercharges are 0).

References

[Hee15] H. v. Hees, Grundlagen der Quantentheorie, I. Teil: Nichtrelativistische Quantentheorie (2015), https://itp.uni-frankfurt.de/~hees/faq-pdf/quant.pdf.