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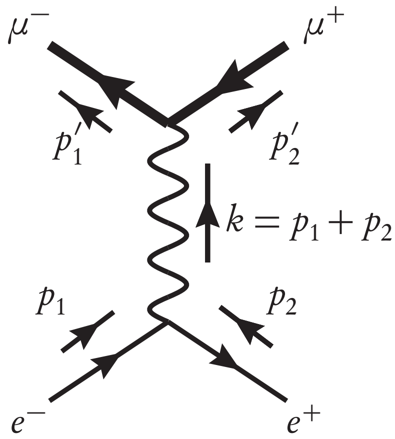
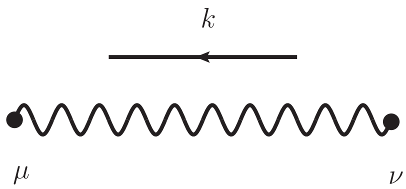
Exercise Sheet 10 – Solutions

Cross Section for $e^+ + e^- \to \mu^+ + \mu^-$

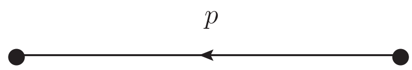
In this exercise we want to calculate the invariant matrix element for the unpolarized cross section for the annihilation of an electron-positron pair to a muon-antimuon pair in leading order QED perturbation theory. One has to evaluate just one tree-level diagram (at order $q^2 = e^2$):



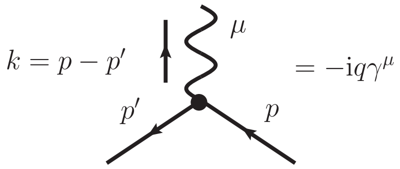
The Feynman rules read



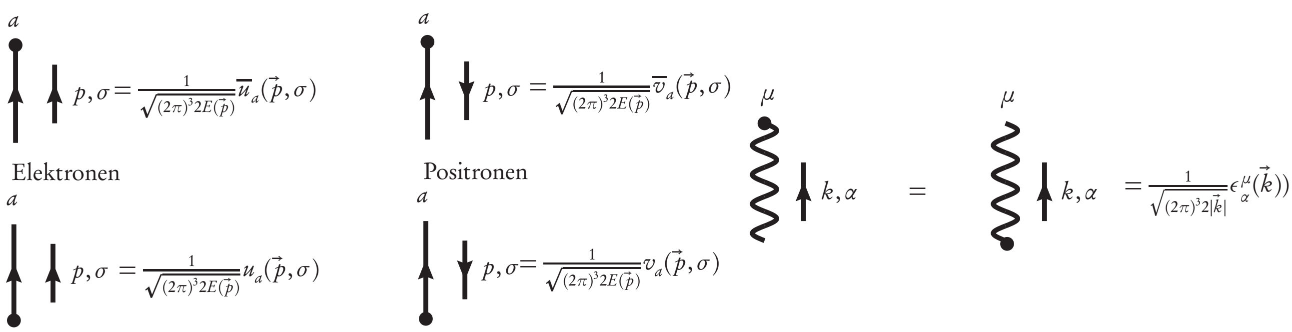
$$= \mathrm{i}\tilde{\Delta}^{\mu\nu}_{\mathrm{Feyn}}(k) = -\frac{\mathrm{i}}{k^2 + \mathrm{i}0^+}$$



$$= \mathrm{i}\tilde{S}_{\mathrm{F}}(p) = \frac{\mathrm{i}(\not{p} + m)}{p^2 - m^2 + \mathrm{i}0^+}$$



$$= -\mathrm{i}q\gamma^\mu$$



The vertex for electrons/positrons and muons/antimuons are the same. In the fermion propagators and $u_\sigma(p)$ and $v_\sigma(p)$ the only difference is the mass, i.e., one has to set $m = m_{\mathrm{e}}$ or $m = m_{\mu}$, corresponding to the involved particle.

(a) Evaluate $\mathrm{i}\mathscr{M}_{fi}$ for definite spins σ_1, σ_2 for the electron and positron in the incoming and σ_1' and σ_2' for the muon and antimuon in the outgoing state.

Solution: It's a diagram of 2nd order. So we have a factor 1/2! from the expansion of the exponential $\mathscr{T}\exp(-\mathrm{i}\int \mathrm{d}^4 x \mathscr{H}_{\mathrm{I}})$, but we also have two different vertices (one for electrons/positrons the other for muons/antimuons and photons). So there's another factor $\binom{2}{1}$ from squaring the sum of the two contributions to $\mathscr{H}_{\mathrm{I}}$. So there's no overall combinatoric factor since also to connect the external lines and the two vertices with the photon-propagator line there's only one possibility.

Thus we finally get

$$\begin{aligned} \mathrm{i}\mathscr{M}_{fi} &= -\mathrm{i}q\overline{u}(\mu,\vec{p}_1{}',\sigma_1')\gamma^\mu v(\mu,\vec{p}_2{}',\sigma_2')\frac{-\mathrm{i}\eta_{\mu\nu}}{(p_1+p_2)^2+\mathrm{i}0^+}(-\mathrm{i}q)\overline{v}(e,\vec{p}_2,\sigma_2)\gamma^\nu u(e,\vec{p}_1,\sigma_1) \\ &= \mathrm{i}q^2\overline{u}(\mu,\vec{p}_1{}',\sigma_1')\gamma^\mu v(\mu,\vec{p}_2{}',\sigma_2')\overline{v}(e,\vec{p}_2,\sigma_2)\gamma^\nu u(e,\vec{p}_1,\sigma_1)\frac{\eta_{\mu\nu}}{s}, \end{aligned} \tag{1}$$

with the Mandelstam variable $s = (p_1+p_2)^2 = (p_1'+p_2')^2$. We note that we can omit the $\mathrm{i}0^+$ in the denominator, because for kinematical reasons, $s \geq 4m_\mu^2$, which we shall discuss below.

(b) Evaluate $|\mathscr{M}_{fi}|^2$. To that end show that

$$\left[\overline{u}_\sigma(\vec{p})\gamma^\mu v_{\sigma'}(\vec{p}\,')\right]^* = \overline{v}_{\sigma'}(\vec{p}\,')\gamma^\mu u_\sigma(\vec{p}). \tag{2}$$

Solution: To take the complex conjugate we can as well as take the adjoint of the matrix-vector-product expression:

$$\begin{aligned} \left[\overline{u}_\sigma(\vec{p})\gamma^\mu v_{\sigma'}(\vec{p}\,')\right]^* &= v_{\sigma'}^\dagger(\vec{p}\,')\gamma^{\mu\dagger}\overline{u}_\sigma(\vec{p})^\dagger \\ &= \overline{v}_{\sigma'}(\vec{p}\,')\gamma^0\gamma^{\mu\dagger}\gamma^{0\dagger}u_\sigma(\vec{p}) \\ &= \overline{v}_{\sigma'}(\vec{p}\,')\gamma^\mu u_\sigma(\vec{p}), \end{aligned} \tag{3}$$

where we have used the standard rules for the γ-matrices (see Lect. 7). Using this for $\mathscr{M}_{fi}$, we find

$$\begin{aligned} |\mathscr{M}_{fi}|^2 = \frac{q^4}{s^2}\eta_{\mu\nu}\eta_{\rho\sigma}\overline{u}(\mu,\vec{p}_1{}',\sigma_1')\gamma^\mu v(\mu,\vec{p}_2{}',\sigma_2')\overline{v}(\mu,\vec{p}_2{}',\sigma_2')\gamma^\rho u(\mu,\vec{p}_1{}',\sigma_1') \\ \overline{v}(\mathrm{e},\vec{p}_2,\sigma_2)\gamma^\nu u(\mathrm{e},\vec{p}_1,\sigma_1)\overline{u}(\mathrm{e},\vec{p}_1,\sigma_1)\gamma^\sigma v(\mathrm{e},\vec{p}_2,\sigma_2). \end{aligned} \tag{4}$$

(c) To get the "unpolarized cross section" we have to average over the initial spins and sum over the final spins, i.e., to calculate

$$\overline{|\mathscr{M}_{fi}|^2} = \frac{1}{2\cdot 2}\sum_{\sigma_1,\sigma_2}\sum_{\sigma_1',\sigma_2'}|\mathscr{M}_{fi}|^2. \tag{5}$$

Hint: The spin-sum formulae are (see presentation/notes to Lect. 7)

$$\sum_\sigma u_\sigma(\vec{p})\overline{u}_\sigma(\vec{p}) = \not{p} + m, \quad \sum_\sigma v_\sigma(\vec{p})\overline{v}_\sigma(\vec{p}) = \not{p} - m. \tag{6}$$

The final result is that you get two traces (one for the electron and one for the muon piece). These can be calculated by using the following trace formulae for Dirac-γ matrices:

$$\mathrm{tr}(\gamma^{\mu_1}\gamma^{\mu_2}\cdots\gamma^{\mu_{2j+1}}) = 0 \quad \text{for} \quad j \in \{0,1,2,\ldots\}, \tag{7}$$

$$\mathrm{tr}(\gamma^{\mu_1}\gamma^{\mu_2}) = 4\eta^{\mu_1\mu_2}, \tag{8}$$

$$\mathrm{tr}(\gamma^{\mu_1}\gamma^{\mu_2}\gamma^{\mu_3}\gamma^{\mu_4}) = 4(\eta^{\mu_1\mu_2}\eta^{\mu_3\mu_4} - \eta^{\mu_1\mu_3}\eta^{\mu_2\mu_4} + \eta^{\mu_1\mu_4}\eta^{\mu_2\mu_3}). \tag{9}$$

For proofs see [Hee11].

Solution: We evaluate the sum over σ_1 and σ_2, using (6)

$$\sum_{\sigma_1,\sigma_2} \overline{v}(\mathrm{e},\vec{p}_2,\sigma_2)\gamma^\nu u(\mathrm{e},\vec{p}_1,\sigma_1)\overline{u}(\mathrm{e},\vec{p}_1,\sigma_1)\gamma^\sigma v(\mathrm{e},\vec{p}_2,\sigma_2) = \sum_{\sigma_2} \overline{v}(\mathrm{e},\vec{p}_2,\sigma_2)\gamma^\nu(\not{p}_1 + m_\mathrm{e})\gamma^\sigma v(\mathrm{e},\vec{p}_2,\sigma_2). \tag{10}$$

To also evaluate the sum over σ_2 we note that for an arbitrary 4×4 matrix Γ, using the index calculus for the Dirac indices (Einstein summation convention implied) and then use again (6):

$$\sum_{\sigma_2} \overline{v}(\mathrm{e},\vec{p}_2,\sigma_2)\hat{\Gamma} v(\mathrm{e},\vec{p}_2,\sigma_2) = \sum_{\sigma_2} \overline{v}_a(\mathrm{e},\vec{p}_2,\sigma_2)\Gamma_{ab} v_b(\mathrm{e},\vec{p}_2) = (\not{p}_2 - m_\mathrm{e}\mathbb{1})_{ba}\Gamma_{ab} = \mathrm{tr}[(\not{p}_2 - m_\mathrm{e})\Gamma]. \tag{11}$$

So we get

$$\sum_{\sigma_1,\sigma_2} \overline{v}(\mathrm{e},\vec{p}_2,\sigma_2)\gamma^\nu u(\mathrm{e},\vec{p}_1,\sigma_1)\overline{u}(\mathrm{e},\vec{p}_1,\sigma_1)\gamma^\sigma v(\mathrm{e},\vec{p}_2,\sigma_2) = \mathrm{tr}[\gamma^\nu(\not{p}_1 + m_\mathrm{e})\gamma^\sigma(\not{p}_1 - m_\mathrm{e})]. \tag{12}$$

Multiplying out the matrices and then using (7-9) one finds

$$\mathrm{tr}[\gamma^\nu(\not{p}_1 + m_\mathrm{e})\gamma^\sigma(\not{p}_1 - m_\mathrm{e})] = 4[p_1^\nu p_2^\sigma + p_1^\sigma p_2^\nu - (\underline{p}_1 \cdot \underline{p}_2 + m_\mathrm{e}^2)g^{\nu\sigma}]. \tag{13}$$

The result can be used also for the sums over the muonic spins. Plugging everything in (5) we get, after some simple but tedious algebra, the "unpolarized matrix element",

$$\overline{|\mathcal{M}_{fi}|^2} = \frac{8q^4}{s^2}\left(\underline{p}_1' \cdot \underline{p}_1 \underline{p}_2' \cdot \underline{p}_2 + \underline{p}_1' \cdot \underline{p}_2 \underline{p}_2' \underline{p}_1 + m_\mathrm{e}^2 \underline{p}_1' \cdot \underline{p}_2' + m_\mu^2 \underline{p}_1 \cdot \underline{p}_2 + 2m_\mathrm{e}^2 m_\mu^2\right). \tag{14}$$

(d) Finally express everything in terms of the invariant Mandelstam variables s and t. The three Mandelstam variables are defined by

$$\begin{aligned} s &= (\underline{p}_1 + \underline{p}_2)^2 = (\underline{p}_1' + \underline{p}_2')^2, \quad t = (\underline{p}_1' - \underline{p}_1)^2 = (\underline{p}_2' - \underline{p}_2)^2, \\ u &= (\underline{p}_1' - \underline{p}_2)^2 = (\underline{p}_2' - \underline{p}_1)^2 = 2(m_\mathrm{e}^2 + m_\mu^2) - s - t. \end{aligned} \tag{15}$$

Note that the four-momenta are on-shell, i.e., $\underline{p}_1^2 = \underline{p}_2^2 = m_\mathrm{e}^2$ and $\underline{p}_1'^2 = \underline{p}_2'^2 = m_\mu^2$.

Solution: With the on-shell conditions we find

$$s = 2(m_\mathrm{e}^2 + \underline{p}_1 \cdot \underline{p}_2) = 2(m_\mu^2 + \underline{p}_1' \cdot \underline{p}_2'), \tag{16}$$

$$t = m_\mathrm{e}^2 + m_\mu^2 - 2\underline{p}_1 \cdot \underline{p}_1' = m_\mathrm{e}^2 + m_\mu^2 - 2\underline{p}_2 \cdot \underline{p}_2', \tag{17}$$

$$u = m_\mathrm{e}^2 + m_\mu^2 - 2\underline{p}_1 \cdot \underline{p}_2' = m_\mathrm{e}^2 + m_\mu^2 - 2\underline{p}_2 \cdot \underline{p}_1', \tag{18}$$

Taking the sum of these expressions and using the four-momentum conservation, $\underline{p}_1 + \underline{p}_2 = \underline{p}_1' + \underline{p}_2'$, leads to the relation

$$s + t + u = 2(m_\mathrm{e}^2 + m_\mu^2). \tag{19}$$

Using the Mandelstam variables to express the products of four-momentum vectors in (14) yields, again after some simple but tedious algebra,

$$\overline{|\mathcal{M}_{fi}|^2} = \frac{4q^4}{s^2}\left[\frac{(t - m_\mathrm{e}^2 - m_\mu^2)^2}{2} + \frac{(u - m_\mathrm{e}^2 - m_\mu^2)^2}{2} + (m_\mathrm{e}^2 + m_\mu^2)\right]. \tag{20}$$

Eliminating u with help of (19) finally gives

$$\overline{|\mathcal{M}_{fi}|^2} = \frac{2q^4}{s^2}\left\{s^2 + 2\left[t(t + s - 2m_\mathrm{e}^2 - 2m_\mu^2) + (m_\mathrm{e}^2 + m_\mu^2)^2\right]\right\} \tag{21}$$

For the especially motivated: Extra work

Calculate the invariant differential and total cross section in the center-momentum frame, where $\vec{p}_1 = -\vec{p}_2 = \vec{p}$ and $\vec{p}_1' = -\vec{p}_2' = \vec{p}'$.

Solution: We first consider the cross section for an arbitrary $2 \rightarrow 2$ collision with particles with masses m_1 and m_2 in the initial and m_1' and m_2' in the final state. First we consider the kinematics in the center-momentum frame, where

$$\underline{p}_1 = \begin{pmatrix} E_1 \\ \vec{p} \end{pmatrix}, \quad \underline{p}_2 = \begin{pmatrix} E_2 \\ -\vec{p} \end{pmatrix}, \quad \underline{p}_1' = \begin{pmatrix} E_1' \\ \vec{p}' \end{pmatrix}, \quad \underline{p}_2' = \begin{pmatrix} E_2' \\ -\vec{p}' \end{pmatrix}. \tag{22}$$

The time arguments are the on-shell energies of the particles, i.e., $E_1 = \sqrt{m_1^2 + \vec{p}^2}$, etc.

The next step is to express the energies and $P = |\vec{p}|$ and $P' = |\vec{p}'|$ in terms of the Mandelstam variable s:

$$\begin{aligned} s &= (E_1 + E_2)^2 = m_1^2 + m_2^2 + 2\underline{p}_1 \underline{p}_2 = m_1^2 + m_2^2 + 2(E_1 E_2 + P^2) \\ &= m_1^2 + m_2^2 + 2[E_1(\sqrt{s} - E_1) + P^2] = m_2^2 - m_1^2 + 2E_1\sqrt{s}. \end{aligned} \tag{23}$$

Solving for E_1:

$$E_1 = \frac{s + m_1^2 - m_2^2}{2\sqrt{s}}. \tag{24}$$

To get E_2 we only need to interchange the labels $1 \longleftrightarrow 2$:

$$E_2 = \frac{s + m_2^2 - m_1^2}{2\sqrt{s}}. \tag{25}$$

For the initial-state center-momentum momentum we get from this

$$P^2 = E_1^2 - m_1^2 = \frac{s^2 - 2s(m_1^2 + m_2^2) + (m_1^2 - m_2)^2}{4s} = \frac{[s - (m_1 + m_2)^2][s - (m_1 + m_2)^2]}{4s}. \tag{26}$$

The same relations, of course, hold in the final state. We only have to put primes on the masses, energies, and center-momentum momentum. Thus (26) implies that the scattering process can only happen for $\sqrt{s} \geq \max(m_1 + m_2, m_1' + m_2')$, i.e., in our case only for $\sqrt{s} \geq 2m_{\mu}$. This is understandable since in the center-momentum frame the least energy one has to use to produce the muon-antimuon pair at rest is $2m_{\mu}$.

As shown in Lect. 9 the invariant differential cross section for two particles in the final state is given by

$$\mathrm{d}\sigma = \frac{|\mathcal{M}_{fi}|^2}{4I}(2\pi)^4 \delta^{(4)}(\underline{p}_1 + \underline{p}_2 - \underline{p}_1' - \underline{p}_2') \frac{\mathrm{d}^3\vec{p}_1'}{(2\pi)^3 2E_1'} \frac{\mathrm{d}^3\vec{p}_2'}{(2\pi)^3 2E_2'}. \tag{27}$$

The invariant "current", I, is, after expressing it in terms of s and P in the center-momentum frame, given by

$$I = \sqrt{(\underline{p}_1 \cdot \underline{p}_2)^2 - m_1^2 m_2^2} = \frac{\sqrt{[s - (m_1 + m_2)^2][s - (m_1 + m_2)^2]}}{2} = \sqrt{s} P. \tag{28}$$

The next aim is to integrate out the four-momentum conserving δ distribution in (27). As before, in this formula we consider energy-momentum conservation and on-shell conditions for the four-momenta to be satisfied. Then we only have to take the integrals over the momenta of the particles in the final state together with the δ distribution. First we take the integral over $\vec{p}_2'$, which together with the δ distribution ensures simply three-momentum conservation. The $\vec{p}_1'$ integral we express in spherical coordinates with $\mathrm{d}^3\vec{p}_1' = P'^2 \mathrm{d}P' \mathrm{d}^2\Omega$:

$$\delta^{(4)}(\underline{p}_1 + \underline{p}_2 - \underline{p}_1' - \underline{p}_2') \mathrm{d}^3\vec{p}_1' \mathrm{d}^3\vec{p}_2' = \delta(\underbrace{E_1 + E_2}_{\sqrt{s}} - E_1' - E_2') P'^2 \mathrm{d}P' \mathrm{d}^2\Omega \tag{29}$$

Now we have

$$E_1'^2 = m_1'^2 + P'^2 = E_2'^2 \Rightarrow P'\mathrm{d}P' = E_1'\mathrm{d}E_1 = E_2'\mathrm{d}E_2'. \tag{30}$$

From this we get

$$\mathrm{d}(E_1' + E_2') = \frac{E_1' + E_2'}{E_1'E_2'} P'\mathrm{d}P' = \frac{\sqrt{s}}{E_1'E_2'} P'\mathrm{d}P' \Rightarrow P'^2 = \frac{E_1'E_2'}{\sqrt{s}} P'\mathrm{d}(E_1' + E_2'). \tag{31}$$

Using this and (28) in (27) we finally get the differential cross section,

$$\frac{\mathrm{d}\sigma}{\mathrm{d}^2\Omega} = \frac{|\mathscr{M}_{fi}|^2}{64\pi^2 s} \frac{P'}{P}. \tag{32}$$

Note that P' and P can be expressed in terms of s, using (26) and the corresponding equation for P' (where one just has to use m_1' and m_2' instead of m_1 and m_2).
Further we can relate the Mandelstam variable t to the scattering angle ϑ between the electron and the muon in our process $\mathrm{e}^+ + \mathrm{e}^- \rightarrow \mu^+ + \mu^-$ by

$$t = (\underline{p}_1 - \underline{p}_1') = m_\mathrm{e}^2 + m_\mu^2 - (s - 2PP' \cos\vartheta), \tag{33}$$

where we used that, because of $m_1 = m_2 = m_\mathrm{e}$ and $m_1' = m_2' = m_\mu$, $E_1 = E_2 = E_1' = E_2' = E = \sqrt{s}/2$ and $E_1' = E_2' = E'$. Plugging this in (21), after another tedious algebra, we find, setting $q^2 = e^2 = 4\pi\alpha$ with the Fine-Structure Constant, $\alpha = /137.035\,999\,177(21)$ [N+24]

$$\overline{|\mathscr{M}_{fi}|^2} = \frac{16\pi^2\alpha^2}{s^2}[s(s + 4m_\mathrm{e}^2 + 4m_{\mathrm{e}^2}) + 16P^2P'^2\cos^2\vartheta] \tag{34}$$

For the total cross section we integrate this over the angles, i.e., over the entire unit sphere S

$$\int_S \mathrm{d}^2\Omega \overline{|\mathscr{M}_{fi}|^2} = \frac{64\pi^3\alpha^2}{3s^2}[3s(s + 4m_\mathrm{e}^2 + 4m_\mu^2) + 16P^2P'^2]. \tag{35}$$

In the ultrarelativistic limit, $\sqrt{s} \gg 2m_\mu$ everything simplifies, because then we can set $m_\mathrm{e} = m_\mu = 0$ and thus $P = P' = \sqrt{s}/2$ in the above formulae, yielding

$$\overline{|\mathscr{M}_{fi}|^2} = 16\pi^2\alpha^2(1 + \cos^2\vartheta), \quad \int_S \mathrm{d}^2\Omega \overline{|\mathscr{M}_{fi}|^2} = \frac{256\pi^3\alpha^2}{3}. \tag{36}$$

Using this in (32) and the corresponding integrated form, we find

$$\frac{\mathrm{d}\sigma}{\mathrm{d}^2\Omega} \underset{s \gg 4m_\mu^2}{\cong} \frac{\alpha^2}{4s}(1 + \cos^2\vartheta), \quad \sigma \underset{s \gg 4m_\mu^2}{\cong} \frac{4\pi\alpha^2}{3s}. \tag{37}$$

A comparison with data from the JADE collaboration (DESY) (including also higher-order QED corrections) can be found in [B+85].

Merry Christmas and a Happy New Year!

References

[B+85] W. Bartel et al. (JADE), New results on $e^+e^- \rightarrow \mu^+\mu^-$ from the JADE detector at PETRA, Z. Phys. C **26**, 507 (1985), `https://doi.org/10.1007/BF01551792`.

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