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Exercise Sheet 9

1. Baker-Campbell-Hausdorff-Formel

We want to prove some formulae concerning operator exponential functions.

(a) We start with

$$\exp(\mathbf{A})\mathbf{B}\exp(-\mathbf{A}) = \sum_{n=0}^{\infty} \frac{1}{n!}[\mathbf{A},\mathbf{B}]_n, \quad [\mathbf{A},\mathbf{B}]_0 = \mathbf{B}, \quad [\mathbf{A},\mathbf{B}]_{j+1} = \left[\mathbf{A},[\mathbf{A},\mathbf{B}]_j\right] \tag{1}$$

Hint: calculate the Taylor expansion of

$$\mathbf{F}(z) = \exp(z\mathbf{A})\mathbf{B}\exp(-z\mathbf{A}) \tag{2}$$

around $z = 0$ end then set $z = 1$.

Solution: We have

$$\mathbf{F}(0) = \mathbf{B}, \quad \mathbf{F}'(z) = \exp(z\mathbf{A})[\mathbf{A},\mathbf{B}]\exp(-z\mathbf{A}) \Rightarrow \mathbf{F}'(0) = [\mathbf{A},\mathbf{B}]. \tag{3}$$

For the higher derivatives from this scheme it follows

$$\mathbf{F}^{(j)}(z) = \exp(z\mathbf{A})[\mathbf{A},\mathbf{B}]_j \exp(-z\mathbf{A}), \tag{4}$$

which be proven easily by induction. From this we get

$$\mathbf{F}(z) = \sum_{j=0}^{\infty} \frac{1}{j!}[\mathbf{A},\mathbf{B}]_j z^j, \tag{5}$$

and (1) follows from this by simply setting $z = 1$.

(b) Next, we consider the **Baker-Campbell-Hausdorff formula** Assume that $\mathbf{A}$ and $\mathbf{B}$ are operators, for which

$$[\mathbf{A},[\mathbf{A},\mathbf{B}]] = [[\mathbf{A},\mathbf{B}],\mathbf{B}] = 0 \tag{6}$$

holds. Then

$$\exp(\mathbf{A}+\mathbf{B}) = \exp\mathbf{A}\,\exp\mathbf{B}\,\exp\left(-\frac{1}{2}[\mathbf{A},\mathbf{B}]\right). \tag{7}$$

Hint: First define

$$\mathbf{F}(z) = \exp[z(\mathbf{A}+\mathbf{B})]. \tag{8}$$

and then apply (1) to $\mathbf{F}(z)\mathbf{A}\mathbf{F}^{-1}(z)$ and use the result to manipulate $F'(z)$ in such a way that you can integrate the resulting differential equation for $\mathbf{F}(z)$, using the initial condition $\mathbf{F}(0) = \mathbb{1}$ (and making use of the commutation relations (6)).

Solution: Using (1) yields

$$\mathbf{F}(z)\mathbf{A}\mathbf{F}^{-1}(z) = \mathbf{A} + z[\mathbf{A}+\mathbf{B},\mathbf{A}] = \mathbf{A} - z[\mathbf{A},\mathbf{B}] \Rightarrow \mathbf{F}(z)\mathbf{A} = \mathbf{A}\mathbf{F}(z) - z[\mathbf{A},\mathbf{B}]\mathbf{F}(x). \tag{9}$$

because all the higher commutators vanish due to (6). Taking the derivative of (8) wrt. z leads to

$$\mathbf{F}'(z) = \exp[z(\mathbf{A}+\mathbf{B})](\mathbf{A}+\mathbf{B}) = \mathbf{F}(z)(\mathbf{A}+\mathbf{B}). \tag{10}$$

With (9) we get

$$\mathbf{F}'(z) = \mathbf{A}\mathbf{F}(z) + \mathbf{F}(z)\mathbf{B} - z\,[\mathbf{A},\mathbf{B}]\,\mathbf{F}(x). \tag{11}$$

Since by assumption (6) $[\mathbf{A},\mathbf{B}]$ commutes with both $\mathbf{A}$ and $\mathbf{B}$, this equation can be integrated in a naive way,

$$\mathbf{F}(z) = \exp(z\mathbf{A})\exp(z\mathbf{B})\exp\left(-\frac{z^2}{2}[\mathbf{A},\mathbf{B}]\right), \tag{12}$$

where we also have used the inital condition $\mathbf{F}(0) = \mathbb{1}$. For $z = 1$ we finally get (7).

2. Various two-point-correlation functions of free KG fields

In the following let $\mathbf{\Phi}(\underline{x})$ be a free self-adjoint Klein"=Gordon"=field operator (representing strictly neutral scalar particles). Its mode decomposition reads

$$\begin{aligned}
\mathbf{\Phi}(\underline{x}) &= \mathbf{\Phi}^{(+)}(\underline{x}) + \mathbf{\Phi}^{(-)}(\underline{x}) \quad \text{with} \\
\mathbf{\Phi}^{(+)}(\underline{x}) &= \int_{\mathbb{R}^3} \mathrm{d}^3\vec{p}\sqrt{\frac{1}{(2\pi)^3 2E_p}}\mathbf{a}(\vec{p})\,\exp(-\mathrm{i}\underline{p}\cdot\underline{x})\Big|_{p^0=E_p} \\
&= \int_{\mathbb{R}^3} \mathrm{d}^4 p\sqrt{\frac{2E_p}{(2\pi)^3}}\Theta(p^0)\mathbf{a}(\vec{p})\exp(-\mathrm{i}\underline{p}\cdot\underline{x}), \quad \mathbf{\Phi}^{(-)}(\underline{x}) = \mathbf{\Phi}^{(+)\dagger}(\underline{x}).
\end{aligned} \tag{13}$$

The on-shell energy is defined as $E_p = +\sqrt{\vec{p}^2 + m^2} > 0$ and the annihilation and creation operators fulfill the bosonic commutator relations

$$\left[\mathbf{a}(\vec{p}),\mathbf{a}(\vec{q})\right] = 0, \quad \left[\mathbf{a}(\vec{p}),\mathbf{a}^\dagger(\vec{q})\right] = \delta^{(3)}(\vec{p}-\vec{q}). \tag{14}$$

Further we use the Fourier transform of the Heaviside unit-step function

$$\Theta(t) = \begin{cases} 1 & \text{for} \quad t > 0, \\ 1/2 & \text{for} \quad t = 0, \\ 0 & \text{for} \quad t < 0, \end{cases} \tag{15}$$

which can be calculated as follows:

$$\tilde{\Theta}(p^0) = \int_{\mathbb{R}} \mathrm{d}t\,\Theta(t)\exp(+\mathrm{i}p^0 t) = \lim_{\epsilon\to 0^+}\int_{\mathbb{R}} \mathrm{d}t\,\Theta(t)\exp[\mathrm{i}t(p^0 + \mathrm{i}\epsilon)] = \frac{\mathrm{i}}{p^0 + \mathrm{i}0^+}. \tag{16}$$

Then

$$\Theta(t) = \int_{\mathbb{R}} \frac{\mathrm{d}p^0}{2\pi}\exp(-\mathrm{i}p^0 t)\tilde{\Theta}(p^0). \tag{17}$$

This can be proven by making use of the theorem of residues using the closed time contours as indicated in the following figure:

Further we use

$$\int_{\mathbb{R}} \mathrm{d}p^0 \exp(-\mathrm{i}p^0 t) = 2\pi\delta(t), \quad \int_{\mathbb{R}} \mathrm{d}t\,\exp(\mathrm{i}p^0 t) = (2\pi)\delta(p^0), \tag{18}$$

and the convolution theorem:

$$c(t) := \int_{\mathbb{R}} \mathrm{d}t' a(t-t')b(t') = \int_{\mathbb{R}} b(t-t')a(t') \Leftrightarrow \tilde{c}(p^0) = \tilde{a}(p^0)\tilde{b}(p^0). \tag{19}$$

Further we define for functions $f(\underline{x})$

$$\tilde{f}(\underline{p}) = \int_{\mathbb{R}^4} \mathrm{d}^4 x f(\underline{x})\exp(+\mathrm{i}\underline{x}\cdot\underline{p}) \Leftrightarrow f(\underline{x}) = \int_{\mathbb{R}^4} \frac{\mathrm{d}^4 p}{(2\pi)^4}\tilde{f}(\underline{p})\exp(-\mathrm{i}\underline{x}\cdot\underline{p}). \tag{20}$$

Now evaluate $\tilde{\Delta}(\underline{p})$ for the following invariant two-point vacuum correlation functions, using the convention

$$\Delta(\underline{x}_1, \underline{x}_2) \equiv \Delta(\underline{x}_1 - \underline{x}_2) = \int_{\mathbb{R}^4} \frac{\mathrm{d}^4 \underline{p}}{(2\pi)^4} \tilde{\Delta}(\underline{p}) \exp[-\mathrm{i}\underline{p} \cdot (\underline{x}_1 - \underline{x}_2)]. \tag{21}$$

(a) The Wightman function

$$\mathrm{i}\Delta_+(\underline{x}_1, \underline{x}_2) = \left\langle \Omega \left| \mathbf{\Phi}(\underline{x}_1) \mathbf{\Phi}(\underline{x}_2) \right| \Omega \right\rangle, \tag{22}$$

Solution: Because $\mathbf{a}(\vec{p})|\Omega\rangle = 0$ and $\langle\Omega|\mathbf{a}^\dagger(\vec{p}) = 0$, we have (with $\underline{p}_1^2 = \underline{p}_2^2 = m^2$)

$$\begin{aligned}
\mathrm{i}\Delta_+(\underline{x}_1, \underline{x}_2) &= \left\langle \Omega \left| [\mathbf{\Phi}^{(+)}(\underline{x}_1) + \mathbf{\Phi}^{(-)}(\underline{x}_1)][\mathbf{\Phi}^{(+)}(\underline{x}_2) + \mathbf{\Phi}^{(-)}(\underline{x}_2)] \right| \Omega \right\rangle = \left\langle \Omega \left| \mathbf{\Phi}^{(+)}(\underline{x}_1) \mathbf{\Phi}^{(-)}(\underline{x}_2) \right| \Omega \right\rangle \\
&= \int_{\mathbb{R}^3} \mathrm{d}^3\vec{p}_1 \int_{\mathbb{R}^3} \mathrm{d}^3\vec{p}_2 \frac{1}{(2\pi)^3 \sqrt{2E_1 2E_2}} \left\langle \Omega \left| \mathbf{a}(\vec{p}_1) \mathbf{a}(\vec{p}_2) \right| \Omega \right\rangle \exp[\mathrm{i}(\underline{p}_2 \cdot \underline{x}_2 - \underline{p}_1 \cdot \underline{x}_1)] \\
&= \int_{\mathbb{R}^3} \mathrm{d}^3\vec{p}_1 \int_{\mathbb{R}^3} \mathrm{d}^3\vec{p}_2 \frac{1}{(2\pi)^3 \sqrt{2E_1 2E_2}} \delta^{(3)}(\vec{p}_1 - \vec{p}_2) \exp[\mathrm{i}(\underline{p}_2 \cdot \underline{x}_2 - \underline{p}_1 \cdot \underline{x}_1)] \\
&= \int_{\mathbb{R}^3} \mathrm{d}^3\vec{p}_1 \frac{1}{(2\pi)^3 2E_1} \exp[\mathrm{i}\underline{p}_1 \cdot (\underline{x}_2 - \underline{x}_1)] \\
&= \int_{\mathbb{R}^4} \frac{\mathrm{d}^4 \underline{p}}{(2\pi)^4} 2\pi\Theta(p^0)\delta(\underline{p}^2 - m^2) \exp[-\mathrm{i}\underline{p}_1 \cdot (\underline{x}_1 - \underline{x}_2)]
\end{aligned} \tag{23}$$

This means that

$$\mathrm{i}\tilde{\Delta}_+(\underline{p}) = 2\pi\Theta(p^0)\delta(\underline{p}^2 - m^2) \tag{24}$$

Note: For the other Wightman function,

$$\mathrm{i}\Delta_-(\underline{x}_1, \underline{x}_2) = \left\langle \Omega \left| \mathbf{\Phi}(\underline{x}_2) \mathbf{\Phi}(\underline{x}_1) \right| \Omega \right\rangle \Rightarrow \mathrm{i}\tilde{\Delta}_-(\underline{p})1 = \mathrm{i}\tilde{\Delta}_+(-\underline{p}) = 2\pi\Theta(-p^0)\delta(\underline{p}^2 - m^2). \tag{25}$$

(b) the commutator function (Pauli-Jordan-Schwinger function)

$$\mathrm{i}\Delta(\underline{x}_1, \underline{x}_2) = \left\langle \Omega \left| \left[\mathbf{\Phi}(\underline{x}_1), \mathbf{\Phi}(\underline{x}_2) \right] \right| \Omega \right\rangle \tag{26}$$

Use the Lorentz invariance of this function to prove from the equal-time commutation relations of the fields that $\Delta(\underline{x}_1, \underline{x}_2) = 0$, if $(\underline{x}_1 - \underline{x}_2)^2 < 0$ (microcausality condition).

Solution: From the definition of the Wightman functions, we have

$$\mathrm{i}\Delta(\underline{x}_1, \underline{x}_2) = \mathrm{i}[\Delta_+(\underline{x}_1, \underline{x}_2) - \Delta_-(\underline{x}_1, \underline{x}_2)] = 2\pi\sigma(p^0)\delta(\underline{p}^2 - m^2), \tag{27}$$

where the sign function is defined as

$$\sigma(p^0) = \begin{cases} 1 & \text{for} \quad p^0 > 0, \\ 0 & \text{for} \quad p^0 = 0, \\ -1 & \text{for} \quad p^0 < 0. \end{cases} \tag{28}$$

To investigate the microcausality condition we Fourier transform back to the space-time domain. Setting $\underline{x}_1 - \underline{x}_2 = \underline{x}$ we get

$$\begin{aligned}
\mathrm{i}\Delta(\underline{x}) &= \int_{\mathbb{R}^4} \frac{\mathrm{d}^4 \underline{p}}{(2\pi)^4} 2\pi\sigma(p^0)\delta(\underline{p}^2 - m^2)\exp(-\mathrm{i}\underline{p}\cdot\underline{x}) \\
&= \int_{\mathbb{R}^3} \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3} \frac{1}{2E_p}\left[\exp(-\mathrm{i}tE_p + \mathrm{i}\vec{p}\cdot\vec{x}) - \exp(+\mathrm{i}tE_p + \mathrm{i}\vec{p}\cdot\vec{x})\right] \\
&= \int_{\mathbb{R}^3} \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3} \frac{\exp(\mathrm{i}\vec{p}\cdot\vec{x})}{2E_p}\left[\exp(-\mathrm{i}tE_p) - \exp(+\mathrm{i}tE_p)\right] \\
&= -\mathrm{i}\int_{\mathbb{R}^3} \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3} \frac{\exp(\mathrm{i}\vec{p}\cdot\vec{x})}{E_p}\sin(E_p t).
\end{aligned} \tag{29}$$

For $t = 0$ this obviously vanishes.

The initial form of this equation shows that the commutator function is invariant under proper orthochronous Lorentz transformations. Particularly because of the δ distribution we ensure that the four-momentum is on shell $p^2 = m^2 > 0$, i.e., time like. So the sign of p^0 does not change under any orthochronous Lorentz transformation. Thus for $\hat{\Lambda} \in \mathrm{SO}(1,3)^{\uparrow}$ indeed we find

$$\mathrm{i}\Delta(\hat{\Lambda}\underline{x}) = \int_{\mathbb{R}^4} \frac{\mathrm{d}^4 \underline{p}}{(2\pi)^4} 2\pi\sigma(p^0)\delta(\underline{p}^2 - m^2)\exp(-\mathrm{i}\underline{p}\cdot\hat{\Lambda}\underline{x}). \tag{30}$$

Substitution of $\underline{p} = \Lambda\underline{p}'$ and using $\mathrm{d}^4\underline{p} = \mathrm{d}^4\underline{p}'$, because $\det\Lambda = 1$, we get

$$\begin{aligned}
\mathrm{i}\Delta(\hat{\Lambda}\underline{x}) &= \int_{\mathbb{R}^4} \frac{\mathrm{d}^4 \underline{p}}{(2\pi)^4} 2\pi\sigma(p^0)\delta(\underline{p}'^2 - m^2)\exp(-\mathrm{i}\hat{\Lambda}\underline{p}'\cdot\hat{\Lambda}\underline{x}), \\
&= \int_{\mathbb{R}^4} \frac{\mathrm{d}^4 \underline{p}}{(2\pi)^4} 2\pi\sigma(p^0)\delta(\underline{p}'^2 - m^2)\exp(-\mathrm{i}\underline{p}'\cdot\underline{x}) = \mathrm{i}\Delta(\underline{x}).
\end{aligned} \tag{31}$$

If now $\underline{x}^2 < 0$ (space-like vector) we can always find a boost such that $(\hat{\Lambda}_{\mathrm{B}}\underline{x})^0 = 0$, because with the boost velocity $\vec{v} = v\vec{n}$ and $\gamma = 1/\sqrt{1-v^2}$

$$\hat{\Lambda}_{\mathrm{B}}\begin{pmatrix} t \\ \vec{x}\end{pmatrix} = \hat{\Lambda}_B\gamma\begin{pmatrix} t - v\vec{n}\cdot\vec{x} \\ \vec{x} - v\vec{n}\end{pmatrix} \tag{32}$$

Since $|t| < |\vec{x}|$, because by assumption $\underline{x}^2 < 0$, we can set $\vec{n} = \vec{x}/|\vec{x}|$ and thus $\vec{n}\cdot\vec{x} = |\vec{x}|$, and then

$$v = \frac{t}{|\vec{x}|}, \quad |v| < 1. \tag{33}$$

So indeed $(\hat{\Lambda}_{\mathrm{B}}\underline{x})^0 = 0$ and thus

$$\Delta(\underline{x}) = \Delta(\hat{\Lambda}_{\mathrm{B}}\underline{x}) = 0. \tag{34}$$

This means the field operators commmute if their arguments are space-like separated.

(c) the retarded propagator

$$\mathrm{iD}_{\mathrm{ret}}(\underline{x}_1, \underline{x}_2) = \left\langle \Omega \left| \Theta(t_1 - t_2)\left[\mathbf{\Phi}(\underline{x}_1), \mathbf{\Phi}(\underline{x}_2)\right]\right| \Omega\right\rangle \tag{35}$$

Solution: Since

$$\mathrm{iD}_{\mathrm{ret}}(\underline{x}_1, \underline{x}_2) \equiv \mathrm{iD}_{\mathrm{ret}}(\underline{x}_1 - \underline{x}_2) = \mathrm{i}\Theta(t_1 - t_2)\Delta(\underline{x}_1 - \underline{x}_2), \tag{36}$$

we can use the convolution theorem (19) for the p^0-t Fourier transform (with $\vec{q}=\vec{p}$):

$$\begin{aligned}
\mathrm{i}\tilde{D}_{\text{ret}}(\underline{p}) &= \int_{\mathbb{R}} \mathrm{d}q^0 \tilde{\Theta}(p^0-q^0)\mathrm{i}\tilde{\Delta}(q^0) = \int_{\mathbb{R}} \frac{\mathrm{d}q^0}{2\pi} \frac{\mathrm{i}}{p^0-q^0+\mathrm{i}0^+} \sigma(q^0) 2\pi\delta(\underline{q}^2-m^2) \\
&= \int_{\mathbb{R}} \frac{\mathrm{d}q^0}{2E_p} \frac{\mathrm{i}}{p^0-q^0+\mathrm{i}0^+}\left[\delta(q^0-E_p)-\delta(q^0+E_p)\right] \\
&= \frac{\mathrm{i}}{2E_p}\left(\frac{1}{p^0-E_p+\mathrm{i}0^+}-\frac{1}{p^0+E_p+\mathrm{i}0^+}\right) \\
&= \frac{\mathrm{i}}{[(p^0+\mathrm{i}0^+)^2-E_p^2]} = \frac{\mathrm{i}}{(p^0)^2-E_p^2+\mathrm{i}\sigma(p^0)0^+} = \frac{\mathrm{i}}{\underline{p}^2-m^2+\mathrm{i}\sigma(p^0)0^+}.
\end{aligned} \tag{37}$$

(d) the Feynman propagator

$$\mathrm{i}\mathrm{D}_{\mathrm{F}}(\underline{x}_1,\underline{x}_2) = \left\langle \Omega \middle| \mathcal{T}\mathbf{\Phi}(\underline{x}_1)\mathbf{\Phi}(\underline{x}_2) \middle| \Omega \right\rangle \tag{38}$$

with the time-ordering operator defined as

$$\mathcal{T}\mathbf{\Phi}(\underline{x}_1)\mathbf{\Phi}(\underline{x}_2) = \Theta(t_1-t_2)\mathbf{\Phi}(\underline{x}_1)\mathbf{\Phi}(\underline{x}_2) + \Theta(t_2-t_1)\mathbf{\Phi}(\underline{x}_2)\mathbf{\Phi}(\underline{x}_1). \tag{39}$$

Solution: With the convolution theorem wrt. the p^0-t Fourier transform we find in the same notation as for retarded Green's function,

$$\begin{aligned}
\mathrm{i}\tilde{D}_{\mathrm{F}}(\underline{p}) &= \int_{\mathbb{R}} \frac{\mathrm{d}q^0}{2\pi}\left[\frac{\mathrm{i}}{p^0-q^0+\mathrm{i}0^+}\mathrm{i}\tilde{\Delta}_+(\underline{q}) + \frac{\mathrm{i}}{q^0-p^2+\mathrm{i}0^+}\mathrm{i}\tilde{\Delta}_-(\underline{q})\right] \\
&= \int_{\mathbb{R}} \frac{\mathrm{d}q^0}{2\pi}\left[\frac{2\pi\mathrm{i}\Theta(q^0)}{p^0-q^0+\mathrm{i}0^+} + \frac{2\pi\mathrm{i}\Theta(-q^0)}{q^0-p^0+\mathrm{i}0^+}\right]\delta(\underline{q}^2-m^2) \\
&= \frac{1}{2E_p}\left[\frac{\mathrm{i}}{p^0-E_p+\mathrm{i}0^+} + \frac{\mathrm{i}}{-E_p-p^0+\mathrm{i}0^+}\right] \\
&= \frac{\mathrm{i}}{2E_p}\left[\frac{1}{p^0-E_p+\mathrm{i}0^+} - \frac{1}{p^0+E_p-\mathrm{i}0^+}\right] \\
&= \frac{\mathrm{i}}{(p^0)^2-(E_p-\mathrm{i}_0)^2} = \frac{\mathrm{i}}{\underline{p}^2-m^2+\mathrm{i}0^+}.
\end{aligned} \tag{40}$$

(e) finally prove that Δ_+ and Δ fulfill the free Klein"=Gordon equation,

$$(\Box_1+m^2)\Delta_+(\underline{x}_1,\underline{x}_2) = (\Box_1+m^2)\Delta(\underline{x}_1,\underline{x}_2) = 0 \tag{41}$$

and that D_{ret} as D_{F} are Green's functions of the Klein"=Gordon operator, i.e.,

$$(\Box_1+m^2)\mathrm{D}_{\text{ret}}(\underline{x}_1,\underline{x}_2) = (\Box_1+m^2)\mathrm{D}_{\mathrm{F}}(\underline{x}_1,\underline{x}_2) = -\delta^{(4)}(\underline{x}_1-\underline{x}_2). \tag{42}$$

Solution: (41) follows immediately from the fact that

$$(\Box_1+m^2)\mathbf{\Phi}(\underline{x}_1) = 0. \tag{43}$$

Further

$$\Box_1\mathrm{D}_{\text{ret}}(\underline{x}_1,\underline{x}_2) = \Box_1\Theta(t_1-t_2)\Delta(\underline{x}_1,\underline{x}_2). \tag{44}$$

Now we can use

$$\partial_{t_1}\Theta(t_1-t_2) = \delta(t_1-t_2), \tag{45}$$

from which

$$\partial_{t_1}\Theta(t_1-t_2)\Delta(\underline{x}_1,\underline{x}_2)=\delta(t_1-t_2)\Delta(\underline{x}_1,\underline{x}_2)+\Theta(t_1-t_2)\partial_{t_1}\Delta(\underline{x}_1,\underline{x}_2)=\Theta(t_1-t_2)\partial_{t_1}\Delta(\underline{x}_1,\underline{x}_2). \tag{46}$$

In the last step we have used that $\Delta(\underline{x}_1,\underline{x}_2)=0$ for $t_1=t_2$. Taking another time derivative we get

$$\partial_{t_1}^2\Theta(t_1-t_2)\Delta(\underline{x}_1,\underline{x}_2)=\delta(t_1-t_2)\partial_{t_1}\Delta(\underline{x}_1,\underline{x}_2)+\Theta(t_1-t_2)\partial_{t_1}^2\Delta(\underline{x}_1,\underline{x}_2). \tag{47}$$

Now

$$\delta(t_1-t_2)\partial_{t_1}\Delta(\underline{x}_1,\underline{x}_2)=-\mathrm{i}\delta(t_1-t_2)\left\langle\Omega\left|\left[\partial_{t_1}\mathbf{\Phi}(\underline{x}_1),\mathbf{\Phi}(\underline{x}_2)\right]\right|\Omega\right\rangle=-\delta^{(4)}(\underline{x}_1-\underline{x}_2). \tag{48}$$

In the last step we have made use of the equal-time canonical commutation equation between $\mathbf{\Phi}$ and $\mathbf{\Pi}=\partial_t\mathbf{\Phi}$, i.e.,

$$[\mathbf{\Pi}(t_1,\vec{x}_1),\mathbf{\Phi}(t_1,\vec{x}_2)]=-\mathrm{i}\delta^{(3)}(\vec{x}_1-\vec{x}_2). \tag{49}$$

So finally we have

$$\begin{aligned}(\Box_1+m^2)\mathrm{D}_{\text{ret}}(\underline{x}_1,\underline{x}_2)&=(\partial_{t_1}^2-\Delta_1+m^2)\mathrm{D}_{\text{ret}}(\underline{x}_1,\underline{x}_2)\\&=-\delta^{(4)}(\underline{x}_1-\underline{x}_2)+\Theta(t_1-t_2)(\partial_{t_1}^2-\Delta_1+m^2)\Delta(\underline{x}_1,\underline{x}_2)\\&=-\delta^{(4)}(\underline{x}_1-\underline{x}_2)+\Theta(t_1-t_2)(\Box_1+m^2)\Delta(\underline{x}_1,\underline{x}_2)=-\delta^{(4)}(\underline{x}_1-\underline{x}_2).\end{aligned} \tag{50}$$

For the Feynman Green's function we have

$$\begin{aligned}\mathrm{D}_{\mathrm{F}}(\underline{x}_1,\underline{x}_2)&=\Theta(t_1-t_2)\Delta_+(\underline{x}_1,\underline{x}_2)+\Theta(t_2-t_2)\Delta_-(\underline{x}_1)\\&=\Theta(t_1-t_2)[\Delta(\underline{x}_1,\underline{x}_2)+\Delta_-(\underline{x}_1,\underline{x}_2)+\Theta(t_2-t_2)[\Delta_-(\underline{x}_1)]\\&=\Theta(t_1-t_2)\Delta(\underline{x}_1,\underline{x}_2)+\Delta_-(\underline{x}_1,\underline{x}_2)=\mathrm{D}_{\text{ret}}(\underline{x}_1,\underline{x}_2)+\Delta_-(\underline{x}_1,\underline{x}_2).\end{aligned} \tag{51}$$

Since Δ_- satisfies the free Klein"=Gordon equation and because of (52) we have

$$(\Box_1+m^2)\mathrm{D}_{\mathrm{F}}(\underline{x}_1,\underline{x}_2)=-\delta^{(4)}(\underline{x}_1-\underline{x}_2). \tag{52}$$

Note: The explicit evaluation of the time-ordered Feynman propagator $\tilde{\Delta}_{\mathrm{F}}$ we see that it can be determined in the space-time domain by adding an "infinitesimal negative imaginary part to m^2", i.e., we look for the Green's function $D(\underline{x})$, which fulfills

$$(\Box+m^2-\mathrm{i}0^+)D(\underline{x})=-\delta^{(4)}(\underline{x}). \tag{53}$$

Solving this with a Fourier ansatz,

$$D(\underline{x})=\int_{\mathbb{R}^4}\mathrm{d}^4\underline{p}\frac{1}{(2\pi)^4}\tilde{D}(\underline{p})\exp(-\mathrm{i}\underline{p}\cdot\underline{x}), \tag{54}$$

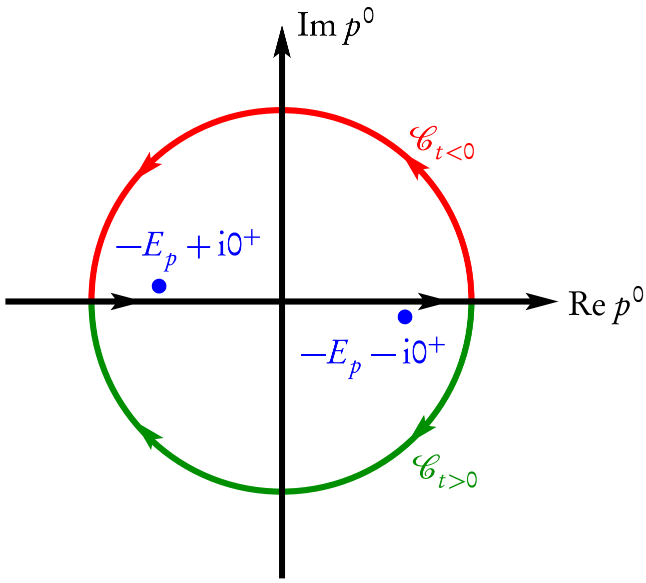
leads to

$$\begin{aligned}(\Box+m^2-\mathrm{i}0^+)D(\underline{x})&=\int_{\mathbb{R}^4}\mathrm{d}^4\underline{p}\frac{1}{(2\pi)^4}(-\underline{p}^2+m^2-\mathrm{i}0^+)\tilde{D}(\underline{p})\exp(-\mathrm{i}\underline{p}\cdot\underline{x})\\&\stackrel{!}{=}-\delta^{(4)}(\underline{x})=-\int_{\mathbb{R}^4}\mathrm{d}^4\underline{p}\frac{1}{(2\pi)^4}\exp(-\mathrm{i}\underline{p}\cdot\underline{x}),\end{aligned} \tag{55}$$

from which

$$\tilde{D}(\underline{p})=\frac{1}{\underline{p}_0^2-m^2+\mathrm{i}0^+}=\tilde{D}_{\mathrm{F}}(\underline{p}). \tag{56}$$

To see the behavior as a function in time, it is sufficient to only do the Fourier transformation with respect to p^0, and this we can calculate using the theorem of residues, using the following two contours in the complex p^0-plane, letting the radii of the semicircles $\rightarrow\infty$, in which limit they do not contribute to the integral, because the integrand is exponentially damped along them, i.e., we indeed calculate the Fourier integral along the entire real axis:



Taking into account that the integration along $\mathscr{C}_{t>0}$ is clockwise and that along $\mathscr{C}_{t<0}$ counter-clockwise around the corresponding poles, the result is

$$\begin{aligned} \mathrm{i}\Delta_{\mathrm{F}}^{(\mathrm{M})}(t,\vec{p}) &= \int_{\mathbb{R}} \frac{\mathrm{d}p^0}{2\pi} \exp(-\mathrm{i}\underline{p}\cdot\underline{x}) \frac{\mathrm{i}}{\underline{p}^2 - m^2 + \mathrm{i}0^+} \\ &= \frac{1}{2E_p}[\Theta(t)\exp[-\mathrm{i}(E_p - \mathrm{i}0^+)t] + \Theta(-t)\exp[-\mathrm{i}(-E_p + \mathrm{i}0^+)t] \xrightarrow[t\to\pm\infty]{} 0. \end{aligned} \tag{57}$$

This means that the Feynman Green's function is the solution for the Klein-Gordon equation with an external source J, which vanishes in the infinite past and future $t \to \pm\infty$.
With this one can drastically shorten the derivation for the generating functional $Z_0[J]$, derived in Lect. 9 using the interaction picture with $\mathbf{H}_{\mathrm{I}} = -J(\underline{x})\mathbf{\Phi}(\underline{x})$.
We start with the Lagrangian

$$\mathscr{L} = \frac{1}{2}(\partial_\mu\Phi)(\partial^\mu\Phi) - \frac{m^2 - \mathrm{i}0^+}{2}\Phi^2 + J\Phi. \tag{58}$$

In the quantized theory the variation principle leads to the *uniquely* solvable equation for the field operator,

$$(-\Box - m^2 + \mathrm{i}0^+)\mathbf{\Phi} = -J, \tag{59}$$

where J is a c-number external source, i.e.,

$$\mathbf{\Phi}(\underline{x}) = -\int_{\mathbb{R}^4} \mathrm{d}^4x' \mathrm{D}_{\mathrm{F}}(\underline{x} - \underline{x}')J(\underline{x}) + \mathbf{\Phi}_0(\underline{x}) = \varphi(\underline{x}) + \mathbf{\Phi}_0(\underline{x}). \tag{60}$$

Here $\mathbf{\Phi}_0(\underline{x})$ is the operatorvalued solution of the free Klein-Gordon equation.
Now we can rewrite the Lagrangian by expanding $\mathbf{\Phi}$ around φ_0. Since $\varphi_0(\underline{x}) \to 0$ by choosing the boundary conditions leading to the use of the Feynman propagator, we can as well consider the action functional and expand it around the c-number solution, φ, i.e., $\mathbf{\Phi} = \varphi + \mathbf{\Phi}'$:

$$S[\mathbf{\Phi}] = \int_{\mathbb{R}^4} \mathrm{d}^4\underline{x}\mathscr{L} = S[\varphi] + \int_{\mathbb{R}^4} \mathrm{d}^4\underline{x} \frac{\delta S[\varphi]}{\delta\varphi(\underline{x})}\mathbf{\Phi}'(\underline{x}) + \int_{\mathbb{R}^4} \mathrm{d}^4\underline{x} \int_{\mathbb{R}^4} \mathrm{d}^4\underline{y} \frac{1}{2} \frac{\delta^2 S[\varphi]}{\delta\varphi(\underline{x})\delta\varphi(\underline{y})}\mathbf{\Phi}'(\underline{x})\mathbf{\Phi}'(\underline{y}). \tag{61}$$

Now the 2nd term vanishes, because φ solves the Lagrange equations of motion, i.e., makes S stationary. The first term reads

$$\begin{aligned} S[\varphi] &= \int_{\mathbb{R}^4} \mathrm{d}^4\underline{x} \left[\frac{1}{2}\left((\partial_\mu\varphi)(\partial^\mu\varphi) - \frac{m^2}{2}\varphi^2\right) + J\varphi\right] \\ &= \int_{\mathbb{R}^4} \mathrm{d}^4\underline{x}\varphi \left[\frac{1}{2}\left(-\Box\varphi - \frac{m^2}{2}\varphi\right) + J\right] \\ &= \frac{1}{2}\int_{\mathbb{R}^4} \mathrm{d}^4\underline{x}\varphi J = -\frac{1}{2}\int_{\mathbb{R}^4} \mathrm{d}^4\underline{x}\int_{\mathbb{R}^4} \mathrm{d}^4 y J(\underline{x})\mathrm{D}_\mathrm{F}(\underline{x}-\underline{y})J(\underline{y}). \end{aligned} \tag{62}$$

The last term in (62) is found from

$$\frac{\delta^2 S[\varphi]}{\delta\varphi(\underline{x})\delta\varphi(\underline{y})} = -2(\Box + m^2)\delta^{(4)}(\underline{x}-\underline{y}), \tag{63}$$

i.e.,

$$\begin{aligned} \int_{\mathbb{R}^4} \mathrm{d}^4\underline{y}\frac{1}{-2}\frac{\delta^2 S[\varphi]}{\delta\varphi(\underline{x})\delta\varphi(\underline{y})}\boldsymbol{\Phi}'(\underline{x})\boldsymbol{\Phi}'(\underline{y}) &= -\frac{1}{2}\int_{\mathbb{R}^4} \mathrm{d}^4\underline{x}\boldsymbol{\Phi}'(\underline{x})(\Box + m^2)\boldsymbol{\Phi}'(\underline{x}) \\ &= \int_{\mathbb{R}^4} \mathrm{d}^4 x\frac{1}{2}\left[(\partial_\mu\boldsymbol{\Phi}'(\underline{x}))(\partial^\mu\boldsymbol{\Phi}'(\underline{x})) - m^2\boldsymbol{\Phi}'^2(\underline{x})\right]. \end{aligned} \tag{64}$$

This is the Lagrangian of a *free* Klein-Gordon field. Together we finally get

$$\mathrm{Z}_0[J] = \mathrm{Z}_0[0]\exp\left[-\frac{\mathrm{i}}{2}\int_{\mathbb{R}^4} \mathrm{d}^4\underline{x}\int_{\mathbb{R}^4} \mathrm{d}^4\underline{y}J(\underline{x})\mathrm{D}_\mathrm{F}(\underline{x}-\underline{y})J(\underline{y})\right]. \tag{65}$$

Since we need this generating funtional anyway only up to a constant factor, because we finally use it to calculate the time-ordered n-Point Green's functions from it, which includes the renormalization factor $\langle\Omega|\mathbf{S}\rangle\Omega = Z[0]$, which finally cancels any constant factor in $\mathrm{Z}_0[J]$.