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Exercise Sheet 8

1. Gauge invariance in QED

Consider the of quantum electrodynamics:

$$\mathscr{L}_{\text{QED}} = \bar{\psi}(\mathrm{i}\not{D} - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}. \tag{1}$$

with $\mathrm{D}_\mu = \partial_\mu + \mathrm{i}qA_\mu$.

(a) Show that $\mathscr{L}_{\text{QED}}$ is invariant under the gauge transformations

$$\begin{aligned} \psi(x) &\mapsto e^{-\mathrm{i}q\alpha(x)}\psi(x), \\ A_\mu(x) &\mapsto A_\mu(x) + \partial_\mu\alpha(x). \end{aligned} \tag{2}$$

Solution: The covariant derivative of the transformed Dirac field reads

$$\mathrm{D}'_\mu\psi' = (\partial_\mu + \mathrm{i}qA_\mu + \mathrm{i}\partial_\mu\alpha)\exp(-\mathrm{i}q\alpha)\psi = \exp(-\mathrm{i}q\alpha)(\partial_\mu - \mathrm{i}q\partial_\mu\alpha + \mathrm{i}qA_\mu \mathrm{i}\partial_\mu\alpha)\psi = \exp(-\mathrm{i}q\alpha)\mathrm{D}_\mu\psi. \tag{3}$$

Since the Dirac-adjoint spinor transforms as

$$\overline{\psi}' = \exp(+\mathrm{i}q\alpha)\overline{\psi}, \tag{4}$$

this implies the invariance of the term in the Lagrangian involving the Dirac field.
Further

$$F'_{\mu\nu} = \partial_\mu A'_\nu - \partial_\nu A'_\mu = \partial_\mu(A_\nu + \partial_\nu\alpha) - \partial_\nu(\partial_\mu A_\nu + \partial_\mu\alpha) = F_{\mu\nu} + (\partial_\mu\partial_\nu - \partial_\nu\partial_\mu)\alpha = F_{\mu\nu}, \tag{5}$$

i.e., the Faraday tensor and thus the kinetic term for the photon field are also gauge invariant.

(b) Show that only adding a mass term $\frac{1}{2}MA^\mu A_\mu$ for the photon breaks gauge invariance.

Solution: The mass term involves

$$A'_\mu A'^\mu = (A_\mu + \partial_\mu\alpha)(A^\mu + \partial^\mu\alpha) = A_\mu A^\mu + 2(\partial_\mu\alpha)A^\mu + (\partial_\mu\alpha)(\partial^\mu\alpha) \neq A_\mu A^\mu, \tag{6}$$

i.e., the term is not gauge invariant.

(c) Show that adding a free real vector field $\theta(x)$ to the theory and adding

$$\mathscr{L}_{\text{Stückel}} = \frac{1}{2}(\partial_\mu\theta)(\partial^\mu\theta) + MA^\mu\partial_\mu\theta \tag{7}$$

to the QED Lagrangian restores gauge invariance despite the mass term for A_μ, if one transforms θ in a clever way.

Solution: We combine the vector-boson mass term with the (7). Then the transformed term reads

$$MA'^\mu\partial'_{\mu\theta} + \frac{M^2}{2}A'_\mu A'^\mu = \frac{M^2}{2}A_\mu A^\mu + M^2(\partial_\mu\alpha)A^\mu + \frac{M^2}{2}(\partial_\mu\alpha)(\partial^\mu\alpha) + \frac{1}{2}(\partial_\mu\theta')(\partial^\mu\theta') + M(A^\mu + \partial^\mu\alpha)\partial_\mu\theta'. \tag{8}$$

To get rid of the red term, comparing with the last term we have to transform the ghost field as

$$\theta' = \theta - M\alpha. \tag{9}$$

Then we get

$$\frac{1}{2}(\partial_\mu \theta')(\partial^\mu \theta') + M(A^\mu + \partial^\mu \alpha)\partial_\mu \theta' = \frac{1}{2}(\partial_\mu \theta - M\partial_\mu \alpha)(\partial^\mu \theta - M\partial^\mu \alpha) + \partial^\mu \alpha)(\partial_\mu - M\partial_\mu \alpha). \tag{10}$$

Using this in (8) we see that indeed finally

$$MA'^\mu \partial_\mu \theta' + \frac{M^2}{2} A'_\mu A'^\mu = MA^\mu \partial_\mu \theta + \frac{M^2}{2} A_\mu A^\mu, \tag{11}$$

i.e., that with the transformation rule (9) for the Stückelberg ghost, the Lagrangian (7) together with the vector-meson mass term becomes gauge invariant.

(d) It is commonly said that gauge symmetries reflect "a redundancy in the mathematical description of the system". Then why do we demand it to be respected in a physical theory?

Solution: A formal reason is that when describing a massless vector field by a field transforming under the (1/2,1/2) representation must be described as a gauge field, i.e., to avoid a continuous amount of polarization states one must assume that with any scalar field α the field $A'_\mu = A_\mu + \partial_\mu \alpha$ mit describe the same physics as A_μ.

Another more phenomenological argument is that classical electrodynamics is successfully describing all electromagnetic phenomena as far as the realm of classical physics is concerned, and thus we are motivated to "quantize Maxwell theory", which in fact is a classical gauge theory in the described sense.

2. Polarizations of the photon

(a) From the QED Lagrangian (1), derive the equations of motion for the ψ and A_μ. How can you connect the latter with the Maxwell's equations $\partial_\mu F^{\mu\nu} = j^\nu$ from classical electrodynamics?

Solution: We consider the action, defined by the QED Lagrangian,

$$S = \int_{V^{(4)}} \mathrm{d}^4 x \left[\overline{\psi}(\mathrm{i}\not{\partial} - q\not{A} - m)\psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right], \quad V^{(4)} = (t_1, t_2) \times \mathbb{R}^3. \tag{12}$$

The equations of motion for the field follows from demanding that the action functional is stationary under arbitrary variations of the independent fields, which vanish at the boundaries of the time interval t_1 and t_2.

Varying with respect to $\overline{\psi}$ (which can be considered as independent of ψ, because ψ consists of 8 independent real field-degrees of freedom (4 complex field-degrees of freedom)):

$$\delta S = \int_{V^{(4)}} \mathrm{d}^4 x \, \delta \overline{\Psi}(\mathrm{i}\not{\partial} - q\not{A} - m)\psi \stackrel{!}{=} 0. \tag{13}$$

Since this must hold true for arbitrary $\delta\overline{\Psi}$ the bracket must vanis, leading to the Dirac equations for a charged particle interacting with an electromagnetic field,

$$(\mathrm{i}\not{\partial} - q\not{A} - m)\psi = 0. \tag{14}$$

Varying ψ leads to the Dirac-adjoint equation of (14), as it should be.

Variation with respect to A^μ leads to

$$\begin{aligned} \delta S &= \int_{V^{(4)}} \mathrm{d}^4 x \left(-\frac{1}{2}\delta F_{\mu\nu} F^{\mu\nu} - q\delta A_\nu \overline{\psi}\gamma^\nu \psi\right) \\ &= \int_{V^{(4)}} \mathrm{d}^4 x \left[-(\partial_\mu \delta A_\nu) F^{\mu\nu} - q\delta A_\nu \overline{\psi}\gamma^\nu\psi\right] \\ &= \int_{V^{(4)}} \mathrm{d}^4 x \delta A_\nu \left(\partial_\mu F^{\mu\nu} - q\overline{\psi}\gamma^\nu\psi\right) \stackrel{!}{=} 0. \end{aligned} \tag{15}$$

This gives the Maxwell equations in relativistic notation,

$$\partial_\mu F^{\mu\nu} = j^\nu = \overline{\psi}\gamma^\mu\psi, \tag{16}$$

with the current conserved due to invariance under *global* U(1) gauge transformations.

(b) The Lorenz gauge fixing condition $\partial_\mu A^\mu = 0$ is *incomplete*, that is, we can still make another transformation

$$A_\mu(x) \mapsto A_\mu(x) + \partial_\mu \Lambda(x). \tag{17}$$

Determine the condition on Λ for this to be true.

(c) Based on your answer to b), choose a convenient gauge parameter to show that physics is unchanged by the transformation

$$\varepsilon^\mu \mapsto \varepsilon^\mu + a k^\mu, \tag{18}$$

for some constant a. In other words, two polarization vectors differing by a multiple of k describe the same free photon. We can use this freedom to set $\varepsilon^0 \equiv 0$. Then, what happens to the Lorenz condition?

Solution: Assuming that $\partial_\mu A^\mu = 0$ and demanding that also $A'_\mu = A_\mu = \partial_\mu \Lambda$ fulfills this condition, yields

$$\partial_\mu A'^\mu = \partial_\mu A^\mu + \Box\Lambda = \Box\Lambda \stackrel{!}{=} 0, \tag{19}$$

i.e., the gauge field Λ must fulfill the source-free wave equation.

(d) The wave function for a *free photon* satisfies the equation

$$\Box A^\mu = 0, \tag{20}$$

which has solutions

$$A^\mu = \varepsilon^\mu(\mathbf{k}) e^{-ik\cdot x}, \quad k^2 = 0. \tag{21}$$

The *polarization vector* ε^μ has 4 components! How can it describe a spin-1 particle?

Solution: From the Lorenz"=gauge condition it follows that

$$k_\mu \epsilon^\mu(\vec{k}) = 0. \tag{22}$$

This is fulfilled for two space-like vectors $\underline{\varepsilon}_j = (0, \vec{\varepsilon}_j)$ with $\vec{\varepsilon}_j \cdot \vec{k} = 0$. One can choose the two linearly independent $\vec{\varepsilon}_j$ as real and such that $\vec{\varepsilon}_1 \times \vec{\varepsilon}_2 = \vec{k}/|\vec{k}|$. These leads to the well-known two transverse wave modes of the electromagnetic field, but another solution of (22) is $\varepsilon_0^\mu = k^\mu$, because of (20) $\underline{k}$ is light-like, i.e., $\underline{k} \cdot \underline{k} = 0$, but any field built by such field modes can just be described by $A_\mu^{(0)} = \partial_\mu \Lambda$, which is gauge equivalent to 0. Because of the Lorenz-gauge condition one must also have $\Box\Lambda = 0$, i.e., for any A_μ we can choose Λ such that for the gauge transformed field $A'_\mu = A_\mu + \partial_\mu \Lambda$ the time component $A'_0 = 0$. Thus for *free em. fields* we can use the residual gauge freedom to demand in addition to the

Lorenz-gauge condition $A^0 = 0$, which is named the radiation gauge, because then we have only the two physical transverse polarization degrees of freedom of the em. field left. Thus we have

$$A^0 = 0, \quad \partial_\mu A^\mu = 0 \Rightarrow \vec{\nabla} \cdot \vec{A} = 0, \tag{23}$$

i.e., we have fulfilled both the Lorenz- and the Coulomb gauge condtion, because of $A^0 = 0$, but note that this is possible only for *free* em. fields, i.e., for $j^\mu = 0$ since for $\rho = j^0 \neq 0$ for the Lorenz"=gauge vector potential one has

$$\Box A_{\mathrm{L}}^0 = \rho \Rightarrow A_{\mathrm{L}}^0 \neq 0. \tag{24}$$

For the Coulomb-gauge potential one gets

$$-\Delta A^0 = \rho \Rightarrow A_{\mathrm{L}}^0 \neq 0. \tag{25}$$

It also turned out that a massless vector field has only *two* (transverse) polarization degrees of freedom and not three, as one would naively expect for a spin-1 field. This is because of the masslessness of the electromagnetic field and the thus necessary gauge invariance, as just discussed.

Physically the two polarization degrees of freedom, defined in terms of intrinsic angular momentum ("spin") are the two possible values of helicity $h = \pm 1$ of the quantized free radiation field, where the helicity is the projection of the field's angular momentum to the direction of its momentum. In contradistinction to the helicity of a massive particle, for massless particles helicity is a Lorentz-invariant quantity, since one cannot "overtake" a photon by any Lorentz boost, because it moves with the speed of light, and no inertial frame can have a relative velocity greater than the speed of light against any other.