Einführung in die Theoretische Kern- und Elementarteilchenphysik 1

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Exercise Sheet 7 – Solutions

Poincaré transformations of the quantized Dirac field

In this exercise we consider the quantized Dirac field and the unitary representation of the Poincré group induced by the field-operator algebra. For the following one needs the generally valid equations for commutators of operator products involving anticommutators:

$$[\mathbf{AB},\mathbf{C}] = \mathbf{A}\{\mathbf{B},\mathbf{C}\} - \{\mathbf{A},\mathbf{C}\}\mathbf{B}. \tag{1}$$

which one proves by just writing out the corresponding operator products explicitly.
We use the simplified Lagrangian

$$\mathscr{L} = \overline{\Psi}(\mathrm{i}\not{\partial} - m)\Psi. \tag{2}$$

From Noether's theorem, applied to space-time translations and boosts (see Lecture 5) one finds the expressions for the corresponding conserved quantities, which are total energy, momentum (forming together the four-vector operator $\underline{\mathbf{P}}$), angular momentum, $\vec{\mathbf{J}}$, and the "boost operators", $\vec{\mathbf{K}}$, (whose conservation says that the "center of energy" moves with constant velocity), written down as normal-ordered operator expressions of the quantized theory:

$$\begin{aligned}
\mathbf{P}_\nu &= \int_{\mathbb{R}^3} \mathrm{d}^3\vec{x} : \boldsymbol{\Theta}^\mu{}_\nu := \int_{\mathbb{R}^3} \mathrm{d}^3\vec{x} : \boldsymbol{\Psi}^\dagger(\underline{x})\mathrm{i}\partial_\nu \boldsymbol{\Psi} : \\
\vec{\mathbf{J}} &= \int_{\mathbb{R}^3} \mathrm{d}^3\vec{x} : \boldsymbol{\Psi}^\dagger \left[\vec{x} \times (-\mathrm{i}\vec{\nabla}) + \hat{\vec{s}}_{\mathrm{D}}\right] \boldsymbol{\Psi} : \\
\vec{\mathbf{K}} &= \int_{\mathbb{R}^3} \mathrm{d}^3\vec{x} : \boldsymbol{\Psi}^\dagger \left[-\vec{x}\mathrm{i}\partial_t - \mathrm{i}t\vec{\nabla} - \gamma^0 \hat{\vec{k}}_{\mathrm{D}}\right] \boldsymbol{\Psi} :
\end{aligned} \tag{3}$$

The colons indicate normal ordering[1], and the field operators obey the Dirac equation,

$$(\mathrm{i}\not{\partial} - m)\boldsymbol{\Psi} = 0, \quad \overline{\boldsymbol{\Psi}}(\mathrm{i}\overleftarrow{\not{\partial}} + m) = 0. \tag{4}$$

The canonical equal-time anticommutator relations read

$$\{\boldsymbol{\Psi}_a(t,\vec{x}), \boldsymbol{\Psi}_b(t,\vec{y})\} = 0, \quad \left\{\boldsymbol{\Psi}_a^\dagger(t,\vec{x}), \boldsymbol{\Psi}_b(t,\vec{y})\right\} = \delta_{ab}\delta^{(3)}(\vec{x}-\vec{y}), \tag{5}$$

and $a, b \in \{1,2,3,4\}$ are labeling the Dirac-spinor components.

(a) Show that $\mathbf{P}_\nu$, $\vec{\mathbf{J}}$, and $\vec{\mathbf{K}}$ are all self-adjoint operators.

Hint: You need to do integration by parts wrt. to the integral over $\vec{x}$. For the time derivatives use the Dirac-equation to express it in terms of spatial derivatives, $\vec{\nabla}$!

Solution: We start with $\mathbf{P}_0 = \mathbf{P}^0 = \mathbf{H}$:

$$\mathbf{H}^\dagger = \int_{\mathbb{R}^3} \mathrm{d}^3\vec{x}(-\mathrm{i}) : [\partial_t \boldsymbol{\Psi}^\dagger(\underline{x})]\boldsymbol{\Psi}(\underline{x}) : \tag{6}$$

[1] Note that when calculating commutators, you don't need to worry about normal ordering, because normal ordering of bilinear operator products only leads to additive (usually diverging) c-number contributions. In this problem you don't need to explicitly consider the mode decomposition of the dirac field in annihilation and creation operators!

To get the time derivative to the field operator $\mathbf{\Psi}$ we follow the hint:

$$-\mathrm{i}\partial_t \mathbf{\Psi}^\dagger = -\mathrm{i}\bar{\psi}\gamma^0 \overleftarrow{\partial_t} = \overline{\psi}(\mathrm{i}\vec{\gamma}\cdot\overleftarrow{\vec{\nabla}} + m) \tag{7}$$

Using this in (6) we obtain

$$\begin{aligned}\mathbf{H}^\dagger &= \int_{\mathbb{R}^3} \mathrm{d}^3\vec{x} : \overline{\mathbf{\Psi}}(m + \mathrm{i}\overleftarrow{\vec{\gamma}\cdot\vec{\nabla}})\mathbf{\Psi} := \int_{\mathbb{R}^3} \mathrm{d}^3\vec{x} : \overline{\mathbf{\Psi}}(m - \mathrm{i}\vec{\gamma}\cdot\vec{\nabla})\mathbf{\Psi} := \int_{\mathbb{R}^3} \mathrm{d}^3\vec{x} : \overline{\mathbf{\Psi}}(\underline{x})\mathrm{i}\gamma^0\partial_t\mathbf{\Psi}(\underline{x}) : \\ &= \int_{\mathbb{R}^3} \mathrm{d}^3\vec{x} : \mathbf{\Psi}^\dagger(\underline{x})\mathrm{i}\partial_t\mathbf{\Psi}(\underline{x}) := \mathbf{H}.\end{aligned} \tag{8}$$

For the momentum it's just integration by parts without any other tricks:

$$\vec{\mathbf{P}}^\dagger = \int_{\mathbb{R}^3} \mathrm{d}^3\vec{x} : (\mathrm{i}\vec{\nabla}\mathbf{\Psi}^\dagger)\mathbf{\Psi} = \int_{\mathbb{R}^3} \mathrm{d}^3\vec{x}\mathbf{\Psi}^\dagger(-\mathrm{i}\vec{\nabla})\mathbf{\Psi} = \vec{\mathbf{P}}. \tag{9}$$

For $\vec{\mathbf{J}}$ we have

$$\begin{aligned}\vec{\mathbf{J}}^\dagger &= \int_{\mathbb{R}^3} \mathrm{d}^3\vec{x} : \mathbf{\Psi}(\underline{x})^\dagger\left[\overleftarrow{\vec{x}\times(+\mathrm{i}\vec{\nabla})} + \vec{\mathbf{s}}_{\mathrm{D}}^\dagger\right]\mathbf{\Psi}(\underline{x}) : \\ &= \int_{\mathbb{R}^3} \mathrm{d}^3\vec{x} : \left[(-\mathrm{i}\vec{\nabla})\mathbf{\Psi}^\dagger \times \vec{x}\mathbf{\Psi} + \mathbf{\Psi}^\dagger\vec{\mathbf{s}}_{\mathrm{D}}\mathbf{\Psi}\right] : \\ &= \int_{\mathbb{R}^3} \mathrm{d}^3\vec{x} : \left[\mathbf{\Psi}^\dagger(+\mathrm{i}\vec{\nabla})\times(\vec{x}\mathbf{\Psi}) + \mathbf{\Psi}^\dagger\vec{\mathbf{s}}_{\mathrm{D}}\mathbf{\Psi}\right] : \\ &= \int_{\mathbb{R}^3} \mathrm{d}^3\vec{x} : \mathbf{\Psi}^\dagger\left[(-\mathrm{i}\vec{x}\times\vec{\nabla}) + \hat{\vec{s}}_{\mathrm{D}}\right]\mathbf{\Psi} :\end{aligned} \tag{10}$$

and finally for $\vec{\mathbf{K}}$

$$\begin{aligned}\vec{\mathbf{K}}^\dagger &= \int_{\mathbb{R}^3} \mathrm{d}^3\vec{x} : \mathbf{\Psi}^\dagger\left[\overleftarrow{-\mathrm{i}\vec{x}\partial_t - \mathrm{i}t\vec{\nabla}} + \frac{\mathrm{i}}{2}\hat{\vec{\gamma}}^\dagger\gamma^0\right]\mathbf{\Psi} : \\ &= \int_{\mathbb{R}^3} \mathrm{d}^3\vec{x} : \mathbf{\Psi}^\dagger\left[[\overleftarrow{-\mathrm{i}\vec{x}\partial_t - \mathrm{i}t\vec{\nabla}} + \gamma^0\frac{\mathrm{i}}{2}\hat{\vec{\gamma}}^\dagger\right]\mathbf{\Psi} :\end{aligned} \tag{11}$$

The only complicated term is the one involving the time derivative acting to the left on $\mathbf{\Psi}^\dagger$, we use again the Dirac equation for $\overline{\mathbf{\Psi}}$ (4):

$$\begin{aligned}-\mathrm{i}\int_{\mathbb{R}^3} \mathrm{d}^3\vec{x} : \left[\mathbf{\Psi}^\dagger\overleftarrow{\partial_t}\vec{x}\mathbf{\Psi}\right] : &= -\mathrm{i}\int_{\mathbb{R}^3} \mathrm{d}^3\vec{x} : \left[\overline{\mathbf{\Psi}}\gamma^0\overleftarrow{\partial_t}\vec{x}\mathbf{\Psi}\right] : \\ &= \int_{\mathbb{R}^3} \mathrm{d}^3\vec{x} : \left[\overline{\mathbf{\Psi}}(\mathrm{i}\hat{\vec{\gamma}}\overleftarrow{\vec{\nabla}} + m)\vec{x}\mathbf{\Psi}\right] : \\ &= \mathrm{i}\int_{\mathbb{R}^3} \mathrm{d}^3\vec{x} : \overline{\mathbf{\Psi}}\left[\gamma^0\vec{x}\partial_t - \vec{\gamma}\right]\mathbf{\Psi} : \\ &= \mathrm{i}\int_{\mathbb{R}^3} \mathrm{d}^3\vec{x} : \mathbf{\Psi}^\dagger\gamma^0\left[\gamma^0\vec{x}\partial_t - \vec{\gamma}\right]\mathbf{\Psi} : \\ &= \mathrm{i}\int_{\mathbb{R}^3} \mathrm{d}^3\vec{x} : \mathbf{\Psi}^\dagger\left[\vec{x}\partial_t - \gamma^0\vec{\gamma}\right]\mathbf{\Psi} :\end{aligned} \tag{12}$$

Using this in (11) and also perform the simple integration by parts on the term with the left-acting $\vec{\nabla}$, we indeed find $\vec{\mathbf{K}}^\dagger = \mathbf{K}$. So despite the fact that the generator of boosts in the Dirac representation of the Lorentz group is not self-adjoint then canonical field-theory formalism guarantees a self-adjoint realization of the entire Poincaré-Lie-algebra, i.e., a unitary representation of this group on the Fock space of the quantum field theory.

(b) The sign conventions for the translation and Lorentz-transformation operators are as follows

$$\begin{aligned}
\mathbf{U}_{\text{transl}}(\underline{a}) &= \exp(\mathrm{i}\underline{a}\cdot\underline{\mathbf{P}}) = \exp(\mathrm{i}a^0\mathbf{P}^0 - \mathrm{i}\vec{a}\cdot\vec{\mathbf{P}}),\\
\mathbf{U}_{\text{boost}}(\eta,\vec{n}) &= \exp(+\mathrm{i}\eta\vec{n}\cdot\vec{\mathbf{K}}),\\
\mathbf{U}_{\text{rot}}(\varphi,\vec{n}) &= \exp(-\mathrm{i}\varphi\vec{n}\cdot\vec{\mathbf{J}}).
\end{aligned} \tag{13}$$

The corresponding transformations for the field operators are as for the corresponding classical fields, i.e.,

$$\begin{aligned}
&\mathbf{U}^{\dagger}_{\text{transl}}(\underline{a})\mathbf{\Psi}(\underline{x})\mathbf{U}_{\text{transl}}(\underline{a}) = \mathbf{\Psi}(\underline{x}-\underline{a}),\\
&\mathbf{U}^{\dagger}_{\text{boost}}(\eta,\vec{n})\mathbf{\Psi}(\underline{x})\mathbf{U}_{\text{boost}}(\eta,\vec{n}) = \exp(+\mathrm{i}\eta\vec{n}\cdot\hat{\vec{k}}_{\text{D}})\mathbf{\Psi}\left(\hat{\Lambda}_{\text{B}}^{-1}(\eta,\vec{n})\underline{x}\right),\\
&\mathbf{U}^{\dagger}_{\text{rot}}(\varphi,\vec{n})\mathbf{\Psi}(\underline{x})\mathbf{U}_{\text{rot}}(\text{D}) = \exp(-\mathrm{i}\varphi\vec{n}\cdot\hat{\vec{s}}_{\text{D}})\mathbf{\Psi}\left(\hat{\Lambda}_{\text{R}}^{-1}(\varphi,\vec{n})\underline{x}\right)
\end{aligned} \tag{14}$$

with

$$\hat{\Lambda}_{\text{B}}^{-1}(\eta,\vec{n}) = \begin{pmatrix} \cosh\eta & -\sinh\eta\vec{n}^{\text{T}} \\ -\sinh\eta\vec{n} & (\cosh\eta-1)\vec{n}\vec{n}^{\text{T}}+\mathbb{1}_3 \end{pmatrix} \tag{15}$$

Expand the both sides of these equations for "infinitesimal" $\delta\underline{a}$, $\delta\eta$, and $\delta\varphi$ to first order in these quantities.

Solution: Expanding the exponential functions of the translation operators to first order in $\delta\underline{a}$ yields

$$\mathbf{U}^{\dagger}_{\text{transl}}(\delta\underline{a})\mathbf{\Psi}(\underline{x})\mathbf{U}_{\text{transl}}(\delta\underline{a}) = \mathbf{\Psi}(\underline{x}) - \mathrm{i}\delta a^{\mu}\left[\mathbf{P}_{\mu},\mathbf{\Psi}(\underline{x})\right]. \tag{16}$$

The right-hand side gives

$$\mathbf{\Psi}(\underline{x}-\delta\underline{a}) = \mathbf{\Psi}(\underline{x}) - \delta a^{\mu}\partial_{\mu}\mathbf{\Psi}(\underline{x}) \Rightarrow -\mathrm{i}\left[\mathbf{P}_{\mu},\mathbf{\Psi}(\underline{x})\right] = -\partial_{\mu}\mathbf{\Psi}(\underline{x}). \tag{17}$$

In the same way one gets for the boosts and rotations

$$\begin{aligned}
-\mathrm{i}\left[\vec{\mathbf{K}},\mathbf{\Psi}(\underline{x})\right] &= (-\vec{x}\partial_t - t\vec{\nabla} + \mathrm{i}\hat{\vec{k}}_{\text{D}})\mathbf{\Psi}(\underline{x}),\\
+\mathrm{i}\left[\vec{\mathbf{J}},\mathbf{\Psi}(\underline{x})\right] &= (-\vec{x}\times\vec{\nabla} - \mathrm{i}\hat{\vec{s}}_{\text{D}})\mathbf{\Psi}(\underline{x}).
\end{aligned} \tag{18}$$

(c) Show that the commutators of the unitary generators (3) resulting from this expansion on the lefthand of the equations (14) side match with what you get on the right-hand side of these equations, i.e., that the self-adjoint operators (3) really are the generators for the corresponding Poincaré transformations.

Hint: For this purpose use the equal-time anticommutator relations for the Dirac field. For anticommutators involving time derivatives of field operators use the Dirac equation to express the time derivatives in terms of spatial derivatives, $\vec{\nabla}$.

Solution: It suffices to show the method for $\mathbf{P}_{\mu}$. In the following we have to introduce an integration variable $\vec{y}$ to evaluate $\mathbf{P}_{\mu}$ as given by (3). We assume that in the four-vectors $\underline{x}$ und $\underline{y}$ the time-arguments are the same, i.e., $t_x = t_y = t$. Then we can use the equal-time commutation relations, using (1). We can also omit the normal-ordering symbol when evaluating the commutator, because they subtract only (diverging) c-number contributions which commute with any operator:

$$\begin{aligned}
[\mathbf{P}_{\nu},\mathbf{\Psi}_a(\underline{x})] &= \int_{\mathbb{R}^3}\mathrm{d}^3\vec{y}\left[\mathbf{\Psi}^{\dagger}_b(\underline{y})\mathrm{i}\partial_{\mu}\mathbf{\Psi}_b(\underline{y}),\mathbf{\Psi}_a(\underline{x})\right]\\
&= \int_{\mathbb{R}^3}\mathrm{d}^3\vec{y}(-\mathrm{i})\left\{\mathbf{\Psi}^{\dagger}_b(\underline{y}),\mathbf{\Psi}_a(\underline{x})\right\}\partial_{\mu}\psi_b(\underline{y})\\
&= -\mathrm{i}\int_{\mathbb{R}^3}\mathrm{d}^3\vec{y}\,\delta^{(3)}(\vec{x}-\vec{y})\delta_{ab}\partial_{\mu}\mathbf{\Psi}_b(\underline{y}) = -\mathrm{i}\partial_{\mu}\mathbf{\Psi}_a(\underline{x}),
\end{aligned} \tag{19}$$

which indeed agrees with (16).

In a completely analogous way one also verifies (18).

Additional remarks

Poincaré-transformation properties of field operators

The Eqs. (a.6) show that $\mathbf{U}^{\dagger}\mathbf{\Psi}(\underline{x})\mathbf{U}$ for translations, boosts, and rotations realize these transformations as unitary representations on Fock space in such a way that the field operators transform as their unquantized ("classical") counter parts, where for **translations**

$$\underline{x}' = \underline{x} + \underline{a}, \quad \Psi'(\underline{x}') = \Psi(\underline{x}) = \Psi(\underline{x}' - \underline{a}), \tag{20}$$

for **boosts**

$$\begin{aligned} \underline{x}' &= \hat{B}(\eta, \vec{n})\underline{x} = \begin{pmatrix} \cosh\eta t + \sinh\eta \vec{n}\cdot\vec{x} \\ \sinh\eta\vec{n}t + \vec{x} + (\cosh\eta - 1)\vec{n}(\vec{n}\cdot\vec{x}) \end{pmatrix}, \\ \Psi'(\underline{x}') &= D_{B,\mathrm{D}}(\eta,\vec{n})\Psi(\underline{x}) = D_{B,\mathrm{D}}(\eta,\vec{n})\Psi\left(\hat{B}^{-1}(\eta,\vec{n})\underline{x}'\right), \end{aligned} \tag{21}$$

and **rotations**

$$\begin{aligned} \underline{x}' &= \hat{R}(\varphi, \vec{n})\underline{x} = \begin{pmatrix} t \\ (\vec{n}\cdot\vec{x})\vec{n} + (\vec{n}\times\vec{x})\times\vec{n}\cos\varphi + \vec{n}\times\vec{x}\sin\varphi \end{pmatrix}, \\ \Psi'(\underline{x}') &= D_{B,\mathrm{D}}(\eta,\vec{n})\Psi(\underline{x}) = D_{B,\mathrm{D}}(\eta,\vec{n})\Psi\left(\hat{B}^{-1}(\eta,\vec{n})\underline{x}'\right). \end{aligned} \tag{22}$$

So (14) describes the transformation from $\psi(\underline{x})$ to $\psi'(\underline{x}')$, writing again $\underline{x}$ instead of $\underline{x}'$.

Since any translation in Minkowski spacetime as well as proper orthochronous Lorentz transformations can be built from these transformations, together these build a unitary representation of the **proper orthochronous Poincaré group**, which is the part of the symmetry group of four-dimensional Minkowski space as an affine point space, where the "points" are physically to be interpreted as "events". This group is called $\mathrm{ISO}(1,3)^{\uparrow}$, i.e., the translations and Lorentz transformations as "isometries" since all the transformations describe the parallel transport of vectors $\overrightarrow{AB}$ as well as "pseudo-rotations" of such vectors around their initial point, A.

All this is in complete analogy to the usual three-dimensional Euclidean affine point space, defining the Euclidean space in the usual geometric sense. There the isometries are also the translations as well as rotations of vectors like $\overrightarrow{AB}$.

Physical meaning of the field operators

The physical meaning of the field operators $\mathbf{\Psi}(\underline{x})$ and $\mathbf{\Psi}^{\dagger}(\underline{x})$ are a bit less intuitive than in non-relativistic field theory, where they are simply the annihilation and creation operators for position-spin eigenstate $|\vec{x}, s\rangle$.

In our relativistic theory we have to take superpositions of both positive-frequency and negative-frequency eigenmodes in order to realize the "local Poincaré-transformation" behavior as discussed above, and in order to have a Hamiltonian, whose spectrum is positive semidefinite (or at least bounded from below), guaranteeing the existence of a ground state of lowest energy and thus the stability of the many-particle system, in the mode decomposition we have use annihilation operators together with the positive-frequency modes and creation operators together with the negative-frequence modes.

This means that when applying a $\mathbf{\Psi}(\underline{x})$ to a Fock state with total charge Q, we create a state, which carries a charge $Q - q$, which is realized as a superposition of annihilating a "particle" (carrying charge $+q$) and creating an "anti-particle" (carrying a charge $-q$). Correspondingly $\mathbf{\Psi}^\dagger(\underline{x})$ leads to a state with charge $Q+q$ (superposition of creating a particle or annihilating an anti-particle) at the space-time point $\underline{x}$.

Correspondingly the transformations written in (14), describe the corresponding lowering and raising of charge in reference frames, which are translated, boosted, or rotated relative to the original one. That these events are completely invariantly described in any such inertial frame of reference is the manifestation of the special-relativistic realization of the indistinguishability of all inertial reference frames, i.e., two experiments lead to the same result, when they only differ by the time when they are performed or by a translation or a rotation of the measurement device or if they are fixed in either inertial reference frame. Formally that's described by the invariance of the physical laws under proper orthocrhonous Poincaré transformations, i.e., the symmetry group of affine Minkowski spacetime.