Institut für Theoretische Physik Goethe
Universität Frankfurt
WiSe 2025/2026
PD Dr. Hendrik van Hees

Exercise Sheet 12

Problem 1: Spontaneously breaking discrete and continous symmetries

Consider the following Lagrangian

$$\begin{aligned}\mathscr{L} &= \frac{1}{2}(\partial_\mu \phi)^2 - V(\phi) \\ &= \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}\mu^2\phi^2 - \frac{1}{4}\lambda\phi^4\end{aligned} \tag{1}$$

that describes a real scalar field ϕ with a specific potential term. It is straightforward to show that $\mathscr{L}$ is invariant under $\phi \to -\phi$. Regarding the parameters (λ, μ^2), λ should be positive to ensure the existence of a ground state but, in principle, μ^2 could be positive or negative. The minimum of $V(\phi)$, ϕ_0, turns out to be:

$$\phi_0 = \begin{cases} 0 & \text{for} \quad \mu^2 > 0 \\ \pm\sqrt{\frac{-\mu^2}{\lambda}} \equiv v & \text{for} \quad \mu^2 < 0 \end{cases} \tag{2}$$

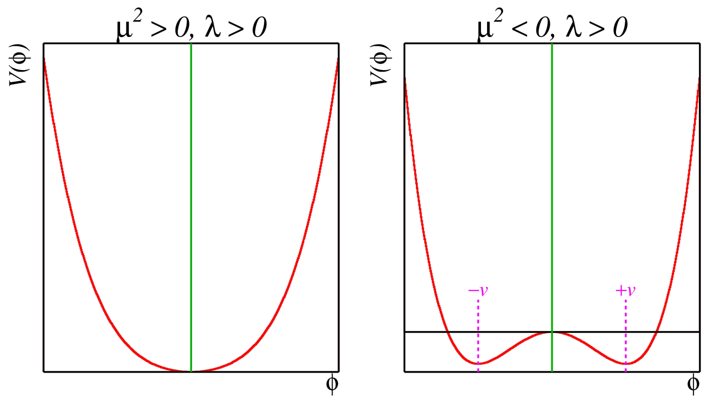

Figure 1: $V(\phi)$ as given by Eq. 1 in two different cases: $\mu^2 > 0$ (left) and $\mu^2 < 0$ (right).

The potential, $V(\phi)$, in both scenarios is shown in Fig. 1. For $\mu^2 > 0$ the quantum field must have a null vacuum expectation value (VEV) . This is not the case for ϕ when $\mu^2 < 0$

$$\langle 0|\phi|0\rangle = v \tag{3}$$

i.e. the vacuum is degenerate. Thus, a field η that is centered at the vacuum should be introduced

$$\eta = \phi - v \tag{4}$$

such that this shifted field satisfies

$$\langle 0|\eta|0\rangle = 0. \tag{5}$$

Then, at the quantum level, the same system is described by $\eta(x)$ with Lagrangian

$$\mathscr{L} = \frac{1}{2}(\partial_\mu \eta)(\partial^\mu \eta) - \lambda v^2\eta^2 - \lambda v\eta^3 - \frac{\lambda}{4}\eta^4 \tag{6}$$

that describes a massive scalar field ($m_\eta = \sqrt{2\lambda}v$) but is no longer invariant under $\eta \to -\eta$. $\mathscr{L}(\phi)$ had the symmetry but the parameters (λ, μ^2) can be such that the ground state of the Hamiltonian is not symmetric: the symmetry has been spontaneously broken.

The same procedure can be applied to more complex scenarios.

Consider a complex scalar field $\phi(x)$ with Lagrangian

$$\mathscr{L} = (\partial_\mu\phi^*)(\partial^\mu\phi) - V(|\phi|) = (\partial_\mu\phi^*)(\partial^\mu\phi) - \mu^2|\phi|^2 - \lambda|\phi|^4 \tag{7}$$

Show that it is invariant under a global U(1) symmetry, i.e., $\phi \to \exp(-\mathrm{i}\alpha)\phi$, $\alpha = \text{const} \in \mathbb{R}$. Write down the Lagrangian in terms of (ϕ_1,ϕ_2) where

$$\phi = \frac{1}{\sqrt{2}}(\phi_1 + \mathrm{i}\phi_2) \tag{8}$$

Repeat the previous procedure in the case of the discrete symmetry to compute ϕ_0. Write down the Lagrangian in terms of the shifted fields:

$$\begin{aligned} \eta &= \phi_1 - v \\ \xi &= \phi_2 \end{aligned} \tag{9}$$

How many massive and massless scalar particles does it contain? What does this fact have to do with the broken symmetry?

Hint: If we repeat the same calculation but with an SU(2) triplet instead of a scalar, the Lagrangian in terms of the shifted fields would no longer be invariant under SU(2) but under U(1) and it would contain 2 massless scalar fields.

Problem 2: Spontaneously breaking gauge invariance

Consider a U(1) gauge-invariant Lagrangian for a complex scalar field $\phi(x)$

$$\mathscr{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (\mathrm{D}_\mu\phi)^*(D^\mu\phi)\underbrace{-\mu^2\phi^*\phi - \lambda(\phi^*\phi)^2}_{-V(|\phi|)} \tag{10}$$

with the gauge-covariant derivative

$$\mathrm{D}_\mu = \partial_\mu + \mathrm{i}qA_\mu(\underline{x}) \tag{11}$$

and the field-strength tensor

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{12}$$

that is invariant under

$$\begin{aligned} \phi'(\underline{x})\phi(x) &= \exp[-\mathrm{i}q\Theta(\underline{x})]\phi(\underline{x}) \\ A'_\mu(\underline{x}) &= A_\mu(x) + \partial_\mu\Theta(x) \end{aligned} \tag{13}$$

Compute the minimum $\phi_0 = v/\sqrt{2} > 0 \in \mathbb{R}$ of the potential of $V(|\phi|)$ for $\mu^2 < 0$. In this case instead of writing ϕ in terms of the shifted fields (η, ξ) as

$$\phi(x) \equiv \frac{1}{\sqrt{2}}[v + \eta(x) + \mathrm{i}\xi(x)], \quad \langle 0|\eta|0\rangle = \langle 0|\xi|0\rangle = 0 \tag{14}$$

it is more transparent to write

$$\phi(x) \equiv \exp(-\mathrm{i}q\xi(x)/v)\frac{1}{\sqrt{2}}[v + \eta(x)], \quad \langle 0|\eta|0\rangle = \langle 0|\xi|0\rangle = 0 \tag{15}$$

and exploit local gauge invariance so that

$$\phi'(x) = \exp(\mathrm{i}q\xi(x)/v)\phi(x) = \frac{1}{\sqrt{2}}[v + \eta(x)] \tag{16}$$

that is equivalent to choose the gauge field $\Theta(x) = -\xi(x)/v$. This choice is called unitary gauge. Rewrite $\mathscr{L}$ in terms of $\eta(x)$ and analyse the different terms that it contains. Is the gauge boson massless?

Discuss, how the various "field-degrees of freedom" are redistributed when comparing the cases with and without broken *local* gauge invariance, and discuss, why there are no massless Goldstone modes in this case in contradistinction to the case of a spontaneously broken *global* symmetry, as considered in Problem 1.

Solution: It's easy to see that again the parameters in the potential and the normalization of ϕ are chosen such that $v = \sqrt{-\mu^2/\lambda} > 0 \in \mathbb{R}$ (for $\mu^2 < 0$).

The gauge-transformed gauge field is

$$A'_\mu = A_\mu + \partial_\mu \Theta = A_\mu - \frac{1}{v}\partial_\mu \xi. \tag{17}$$

The Lagrangian in the new fields is

$$\mathscr{L} = -\frac{1}{4}F'_{\mu\nu}F'^{\mu\nu} + \frac{1}{2}[\mathrm{D}'_\mu(v+\eta)]^*[\mathrm{D}'_\mu(v+\eta)] - V(|v+\eta|/2) \tag{18}$$

As to be expected, since $\mathscr{L}$ is invariant under the *local* gauge transformations, the field ξ does