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Exercise Sheet 11

Problem 1: Color SU(3)

We consider color states for quarks $|c\rangle$ ($c \in \{r, g, b\}$ and anti-quarks $|\bar{c}\rangle$ ($c \in \{\overline{r}, \overline{g}, \overline{b}\}$). They transform under the fundamental 3-representation and its conjugate complex 3*-representation respectively, i.e., for any $\hat{U} \in \mathrm{SU}(3)$

$$\left|c'\right\rangle = \sum_c |c\rangle\, U_{cc'}, \quad \left|\bar{c}'\right\rangle = \sum_{c=1}^{3} |\bar{c}\rangle\, U^*_{cc'}. \tag{1}$$

Hadrons must be color singlets, i.e., they must be states that do not change under color-SU(3) transformations. Show that...

(a) ...for a meson, i.e., a bound state of a quark and an antiquark, the color state

$$\left|\psi_{c,\text{meson}}\right\rangle = \sum_{c_1,c_2} C_{c_1 c_2} \left|\bar{c}_1, c_2\right\rangle \tag{2}$$

is determined by $C_{c_1 c_2} = \delta_{c_1 c_2}$.

(b) ...for a baryon, i.e., a bound state of three quarks, the color state is the totally antisymmetric state

$$\left|\psi_{c,\text{baryon}}\right\rangle = \sum_{c_1,c_2,c_3} \epsilon_{c_1 c_2 c_3} \left|c_1, c_2, c_2\right\rangle, \tag{3}$$

where ϵ_{jkl} is the usual totally antisymmetric Levi-Civita symbol with $\epsilon_{rgb} = 1$.

(c) Consider the three antisymmetrized “di-quark”color state

$$\left|\psi_{c_1}\right\rangle = \sum_{c_2,c_3} \epsilon_{c_1 c_2 c_3} \left|c_2, c_3\right\rangle. \tag{4}$$

Show that it transforms under the conjugate-complex 3* representation of the color SU(3).

Hint: Note that $\det \hat{U} = 1$, which implies that

$$\sum_{j,k,l} \epsilon_{jkl} U_{jj'} U_{kk'} U_{ll'} = \sum_{j,k,l} \epsilon_{jkl} U_{j'j} U_{k'k} U_{l'l} = \det \hat{U} \epsilon_{j'k'l'}. \tag{5}$$

Problem 2: Some isospin gymnastics

In this exercise we want to investigate the representation theory of SU(2) (in the physical interpreataion as “isospin”), using Schwinger’s harmonic-oscillator approach, corresponding to a two-flavor-quark model (describing the two lightest u- and d-Quarks).
A symmetric harmonic oscillator in 2 dimensions can be described by the Hamiltonian

$$\mathbf{H} = \omega \mathbf{N}, \quad \mathbf{N} = \mathbf{N}_1 + \mathbf{N}_2, \quad \mathbf{N}_f = \mathbf{a}_f^\dagger \mathbf{a}_f, \quad f \in \{1, 2\}. \tag{6}$$

where the $\mathbf{a}_f$ obey the commutation relations of bosonic annihilation operators of a field mode, which is the connection with QFT:

$$\left[\mathbf{a}_f, \mathbf{a}_g\right] = 0, \quad \left[\mathbf{a}_f, \mathbf{a}_g^\dagger\right] = \delta_{fg} \mathbb{1}. \tag{7}$$

For the energy-eigenvalue problem we can choose the flavor-number operators $\mathbf{N}_f$ as a complete set of compatible observables, and from the quantum mechanics of harmonic oscillators we know that the corresponding complete orthonormal set of energy eigenstates can be constructed from the "vacuum" $|\Omega\rangle = |N_1 = 0, N_2 = 0\rangle$, defined by

$$\mathbf{a}_f |\Omega\rangle = 0 \tag{8}$$

for all f and

$$|N_1, N_2\rangle = \prod_{f=1}^{2} \frac{1}{\sqrt{N_f!}} \mathbf{a}_f^{\dagger N_f} |\Omega\rangle. \tag{9}$$

It is now clear that the harmonic oscillator has SU(2) as a symmetry group, i.e., the Hamiltonian is invariant under the transformation

$$\mathbf{a}'_f = \sum_{g=1}^{2} U_{fg} \mathbf{a}_g \tag{10}$$

with $\hat{U} \in \mathrm{SU}(2)$, which implies that

$$\mathbf{a}'^\dagger_f = \sum_{g=1}^{2} \mathbf{a}_g^\dagger U_{fg}^* = \sum_{g=1}^{2} \mathbf{a}_g^\dagger (\hat{U}^\dagger)_{gf}. \tag{11}$$

We want to show that the familiar irreducible spin represenations with $s \in \{0, 1/2, \ldots\}$ can be constructed by the action of creation and annihilation operators on the complete set of orthonormal eigenfunctions $|N_1, N_2\rangle$ with fixed total number $N = N_1 + N_2$. Obviously the three operators

$$\mathbf{s}_+ = \mathbf{a}_1^\dagger \mathbf{a}_2, \quad \mathbf{s}_- = \mathbf{a}_2^\dagger \mathbf{a}_1 = \mathbf{a}_+^\dagger, \quad \mathbf{s}_3 = \frac{1}{2}(\mathbf{N}_1 - \mathbf{N}_2) \tag{12}$$

leave the eigenspaces of $\mathbf{N}$ with eigenvalues $N \in \{0, 1, 2, \ldots\}$ invariant.

Show that the three operators

$$\mathbf{s}_1 = \frac{1}{2}(\mathbf{s}_+ + \mathbf{s}_-), \quad \mathbf{s}_2 = \frac{1}{2\mathrm{i}}(\mathbf{s}_+ - \mathbf{s}_-), \quad \mathbf{s}_3 \tag{13}$$

fulfill the commutation relations of angular-momentum components

$$\left[\mathbf{s}_j, \mathbf{s}_k\right] = \mathrm{i}\epsilon_{jkl} \mathbf{s}_l. \tag{14}$$

Calculate $\vec{\mathbf{s}}^2$ and show that the operation of this isospin algebra on the eigenspace of $\mathbf{N}$ with eigenvalue N realize the known irreducible representations of the su(2) algebra with $s = N/2$, where $s(s+1)$ are the eigenvalues of $\vec{\mathbf{s}}^2$.

Remark on the conjugate complex fundamental representation

The anti-particle operators by definition transform with $\hat{U}^*$ with generators $-\hat{t}^*$. Show however that this realizes a representation which is unitarily equivalent to the representation with $s = 1/2$.

Hint: Use the known Pauli matrices and show that $-\hat{\vec{\sigma}}^* = \hat{\sigma}_2 \hat{\vec{\sigma}} \hat{\sigma}_2$.

Note: With the same technique one can also treat the representation theory of the SU(N) group for any $N \in \mathbb{N}$. For $N \geq 3$ there are two inequivalent fundamental representations with two distinct kinds of SU(N)-spinors, the one kind transforming with $\hat{U} \in \mathrm{SU}(N)$ and the other with $\hat{U}^*$, i.e., the conjugate complex representation, i.e., for more than 2 flavors the flavor-symmetry operations and the corresponding non-zero hypercharges are different for quarks and antiquarks (for u- and d-quarks all hypercharges are 0).