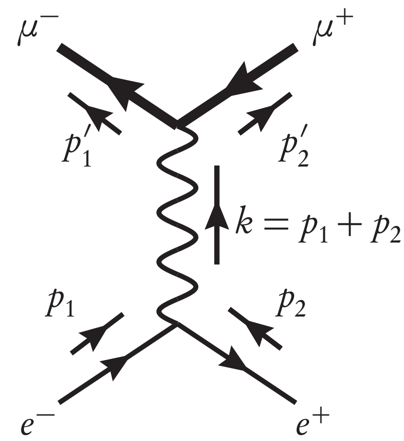
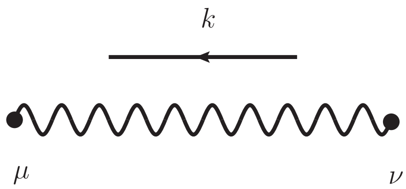
Exercise Sheet 10

Cross Section for $e^+ + e^- \rightarrow \mu^+ + \mu^-$

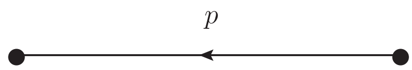
In this exercise we want to calculate the invariant matrix element for the unpolarized cross section for the annihilation of an electron-positron pair to a muon-antimuon pair in leading order QED perturbation theory. One has to evaluate just one tree-level diagram (at order $q^2 = e^2$):



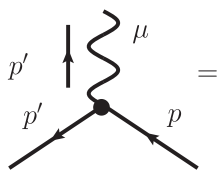
The Feynman rules read



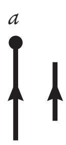
$$= \mathrm{i}\tilde{\Delta}^{\mu\nu}_{\text{Feyn}}(k) = -\frac{\mathrm{i}}{k^2 + \mathrm{i}0^+}$$



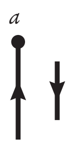
$$= \mathrm{i}\tilde{S}_{\mathrm{F}}(p) = \frac{\mathrm{i}(\not{p} + m)}{p^2 - m^2 + \mathrm{i}0^+}$$



$$= -\mathrm{i}q\gamma^\mu$$



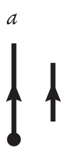
$p, \sigma = \frac{1}{\sqrt{(2\pi)^3 2E(\vec{p})}} \overline{u}_a(\vec{p}, \sigma)$



$p, \sigma = \frac{1}{\sqrt{(2\pi)^3 2E(\vec{p})}} \overline{v}_a(\vec{p}, \sigma)$

Elektronen

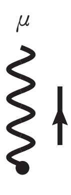
Positronen



$p, \sigma = \frac{1}{\sqrt{(2\pi)^3 2E(\vec{p})}} u_a(\vec{p}, \sigma)$

$p, \sigma = \frac{1}{\sqrt{(2\pi)^3 2E(\vec{p})}} v_a(\vec{p}, \sigma)$

k, α =



$k, \alpha = \frac{1}{\sqrt{(2\pi)^3 2|\vec{k}|}} \epsilon^{\mu}_{\alpha}(\vec{k}))$

The vertex for electrons/positrons and muons/antimuons are the same. In the fermion propgators and $u_\sigma(p)$ and $v_\sigma(p)$ the only difference is the mass, i.e., one has to set $m = m_{\mathrm{e}}$ or $m = m_{\mu}$, corresponding to the involved particle.

(a) Evaluate $\mathrm{i}\mathcal{M}_{fi}$ for definite spins σ_1, σ_2 for the electron and positron in the incoming and σ_1' and σ_2' for the muon and antimuon in the outgoing state.

(b) Evaluate $|\mathcal{M}_{fi}|^2$. To that end show that

$$\left[\overline{u}_\sigma(\vec{p})\gamma^\mu v_{\sigma'}(\vec{p}')\right]^* = \overline{v}_{\sigma'}(\vec{p}')\gamma^\mu u_\sigma(\vec{p}). \tag{1}$$

(c) to get the "unpolarized cross section" we have to average over the initial spins and sum over the final spins, i.e., to calculate

$$\frac{1}{2\cdot 2}\sum_{\sigma_1,\sigma_2}\sum_{\sigma_1',\sigma_2'}|\mathcal{M}_{fi}|^2. \tag{2}$$

Hint: The spin-sum formulae are (see presentation/notes to Lect. 7)

$$\sum_\sigma u_\sigma(\vec{p})\overline{u}_\sigma(\vec{p}) = \not{p} + m, \quad \sum_\sigma v_\sigma(\vec{p})\overline{v}_\sigma(\vec{p}) = \not{p} - m. \tag{3}$$

The final result is that get two traces (one for the electron and one for the muon piece). These can be calculated by using the following trace formulae for Dirac-γ matrices:

$$\begin{aligned}
&\mathrm{tr}(\gamma^{\mu_1}\gamma^{\mu_2}\cdots\gamma^{\mu_{2j+1}}) = 0 \quad \text{for} \quad j \in \{0,1,2,\ldots\},\\
&\mathrm{tr}(\gamma^{\mu_1}\gamma^{\mu_2}) = 4\eta^{\mu_1\mu_2},\\
&\mathrm{tr}(\gamma^{\mu_1}\gamma^{\mu_2}\gamma^{\mu_3}\gamma^{\mu_4}) = 4(\eta^{\mu_1\mu_2}\eta^{\mu_3\mu_4} - \eta^{\mu_1\mu_3}\eta^{\mu_2\mu_4} + \eta^{\mu_1\mu_4}\eta^{\mu_2\mu_3}).
\end{aligned} \tag{4}$$

For proofs see [Hee11].

(d) Finally express everything in terms of the invariant Mandelstam variables s and t. The three Mandelstam variables are defined by

$$\begin{aligned}
&s = (\underline{p}_1 + \underline{p}_2)^2 = (\underline{p}_1' + \underline{p}_2')^2, \quad t = (\underline{p}_1' - \underline{p}_1)^2 = (\underline{p}_2' - \underline{p}_2)^2,\\
&u = (\underline{p}_1' - \underline{p}_2)^2 = (\underline{p}_2' - \underline{p}_1)^2 = 2(m_{\mathrm{e}}^2 + m_{\mu}^2) - s - t.
\end{aligned} \tag{5}$$

Note that the four-momenta are on-shell, i.e., $\underline{p}_1^2 = \underline{p}_2^2 = m_{\mathrm{e}}^2$ and $\underline{p}_1'^2 = \underline{p}_2'^2 = m_{\mu}^2$.

For the especially motivated: Extra work
Calculate the invariant differential and total cross section in the center-momentum frame, where $\vec{p}_1 = -\vec{p}_2 = \vec{p}$ and $\vec{p}_1' = -\vec{p}_2' = \vec{p}'$.

Merry Christmas and a Happy New Year!

References

[Hee11] H. v. Hees, Quantentheorie II (2011), `https://itp.uni-frankfurt.de/~hees/publ/hqm.pdf`.