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Exercise Sheet 9

1. Baker-Campbell-Hausdorff-Formel

We want to prove some formulae concerning operator exponential functions.

(a) We start with

$$\exp(\mathbf{A})\mathbf{B}\exp(-\mathbf{A}) = \sum_{n=0}^{\infty} \frac{1}{n!}[\mathbf{A},\mathbf{B}]_n, \quad [\mathbf{A},\mathbf{B}]_0 = \mathbf{B}, \quad [\mathbf{A},\mathbf{B}]_{j+1} = \left[\mathbf{A},[\mathbf{A},\mathbf{B}]_j\right] \tag{1}$$

Hint: calculate the Taylor expansion of

$$\mathbf{F}(z) = \exp(z\mathbf{A})\mathbf{B}\exp(-z\mathbf{A}) \tag{2}$$

around $z = 0$ end then set $z = 1$.

(b) Next, we consider the **Baker-Campbell-Hausdorff formula** Assume that $\mathbf{A}$ and $\mathbf{B}$ are operators, for which

$$[\mathbf{A},[\mathbf{A},\mathbf{B}]] = [[\mathbf{A},\mathbf{B}],\mathbf{B}] = 0 \tag{3}$$

holds. Then

$$\exp(\mathbf{A}+\mathbf{B}) = \exp\mathbf{A}\,\exp\mathbf{B}\,\exp\left(-\frac{1}{2}[\mathbf{A},\mathbf{B}]\right). \tag{4}$$

Hint: First define

$$\mathbf{F}(z) = \exp[z(\mathbf{A}+\mathbf{B})]. \tag{5}$$

and then apply (1) to $\mathbf{F}(z)\mathbf{A}\mathbf{F}^{-1}(z)$ and use the result to manipulate $F'(z)$ in such a way that you can integrate the resulting differential equation for $\mathbf{F}(z)$, using the initial condition $\mathbf{F}(0) = \mathbb{1}$ (and making use of the commutation relations (3)).

2. Various two-point-correlation functions of free KG fields

In the following let $\mathbf{\Phi}(\underline{x})$ be a free self-adjoint Klein"=Gordon"=field operator (representing strictly neutral scalar particles). Its mode decomposition reads

$$\begin{aligned}
\mathbf{\Phi}(\underline{x}) &= \mathbf{\Phi}^{(+)}(\underline{x}) + \mathbf{\Phi}^{(-)}(\underline{x}) \quad \text{with} \\
\mathbf{\Phi}^{(+)}(\underline{x}) &= \int_{\mathbb{R}^3} \mathrm{d}^3\vec{p} \sqrt{\frac{1}{(2\pi)^3 2E_p}}\mathbf{a}(\vec{p})\,\exp(-\mathrm{i}\underline{p}\cdot\underline{x})\Big|_{p^0=E_p} \\
&= \int_{\mathbb{R}^3} \mathrm{d}^4 p \sqrt{\frac{2E_p}{(2\pi)^3}}\Theta(p^0)\mathbf{a}(\vec{p})\exp(-\mathrm{i}\underline{p}\cdot\underline{x}), \quad \mathbf{\Phi}^{(-)}(\underline{x}) = \mathbf{\Phi}^{(+)\dagger}(\underline{x}).
\end{aligned} \tag{6}$$

The on-shell energy is defined as $E_p = +\sqrt{\vec{p}^2 + m^2} > 0$ and the annihilation and creation operators fulfill the bosonic commutator relations

$$\left[\mathbf{a}(\vec{p}),\mathbf{a}(\vec{q})\right] = 0, \quad \left[\mathbf{a}(\vec{p}),\mathbf{a}^\dagger(\vec{q})\right] = \delta^{(3)}(\vec{p}-\vec{q}). \tag{7}$$

Further we use the Fourier transform of the Heaviside unit-step function

$$\Theta(t) = \begin{cases} 1 & \text{for} \quad t > 0, \\ 1/2 & \text{for} \quad t = 0, \\ 0 & \text{for} \quad t < 0, \end{cases} \tag{8}$$

which can be calculated as follows:

$$\tilde{\Theta}(p^0) = \int_{\mathbb{R}} \mathrm{d}t \Theta(t) \exp(+\mathrm{i}p^0 t) = \lim_{\epsilon \to 0^+} \int_{\mathbb{R}} \mathrm{d}t \Theta(t) \exp[\mathrm{i}t(p^0 + \mathrm{i}\epsilon)] = \frac{\mathrm{i}}{p^0 + \mathrm{i}0^+}. \tag{9}$$

Then

$$\Theta(t) = \int_{\mathbb{R}} \frac{\mathrm{d}p^0}{2\pi} \exp(-\mathrm{i}p^0 t) \tilde{\Theta}(p^0). \tag{10}$$

This can be proven by making use of the theorem of residues using the closed time contours as indicated in the following figure:
Further we use

$$\int_{\mathbb{R}} \mathrm{d}p^0 \exp(-\mathrm{i}p^0 t) = 2\pi\delta(t), \qquad \int_{\mathbb{R}} \mathrm{d}t \exp(\mathrm{i}p^0 t) = (2\pi)\delta(p^0), \tag{11}$$

and the convolution theorem:

$$c(t) := \int_{\mathbb{R}} \mathrm{d}t' a(t - t') b(t') = \int_{\mathbb{R}} b(t - t') a(t') \Leftrightarrow \tilde{c}(p^0) = \tilde{a}(p^0) \tilde{b}(p^0). \tag{12}$$

Further we define for functions $f(\underline{x})$

$$\tilde{f}(\underline{p}) = \int_{\mathbb{R}^4} \mathrm{d}^4 x f(\underline{x}) \exp(+\mathrm{i}\underline{x} \cdot \underline{p}) \Leftrightarrow f(\underline{x}) = \int_{\mathbb{R}^4} \frac{\mathrm{d}^4 p}{(2\pi)^4} \tilde{f}(\underline{p}) \exp(-\mathrm{i}\underline{x} \cdot \underline{p}). \tag{13}$$

Now evaluate $\tilde{\Delta}(\underline{p})$ for the following invariant two-point vacuum correlation functions, using the convention

$$\Delta(\underline{x}_1, \underline{x}_2) \equiv \Delta(\underline{x}_1 - \underline{x}_2) = \int_{\mathbb{R}^4} \frac{\mathrm{d}^4 \underline{p}}{(2\pi)^4} \tilde{\Delta}(\underline{p}) \exp[-\mathrm{i}\underline{p} \cdot (\underline{x}_1 - \underline{x}_2)]. \tag{14}$$

(a) The Wightman function

$$\mathrm{i}\Delta_+(\underline{x}_1, \underline{x}_2) = \left\langle \Omega \middle| \mathbf{\Phi}(\underline{x}_1) \mathbf{\Phi}(\underline{x}_2) \middle| \Omega \right\rangle, \tag{15}$$

(b) the commutator function (Pauli-Jordan-Schwinger function)

$$\mathrm{i}\Delta(\underline{x}_1, \underline{x}_2) = \left\langle \Omega \middle| \left[\mathbf{\Phi}(\underline{x}_1), \mathbf{\Phi}(\underline{x}_2) \right] \middle| \Omega \right\rangle \tag{16}$$

Use the Lorentz invariance of this function to prove from the equal-time commutation relations of the fields that $\Delta(\underline{x}_1, \underline{x}_2) = 0$, if $(\underline{x}_1 - \underline{x}_2)^2 < 0$ (microcausality condition).

(c) the retarded propagator

$$\mathrm{i}\mathrm{D}_{\text{ret}}(\underline{x}_1, \underline{x}_2) = \left\langle \Omega \middle| \Theta(t_1 - t_2) \left[\mathbf{\Phi}(\underline{x}_1), \mathbf{\Phi}(\underline{x}_2) \right] \middle| \Omega \right\rangle \tag{17}$$

(d) the Feynman propagator

$$\mathrm{i}\mathrm{D}_{\mathrm{F}}(\underline{x}_1, \underline{x}_2) = \left\langle \Omega \middle| \mathscr{T} \mathbf{\Phi}(\underline{x}_1) \mathbf{\Phi}(\underline{x}_2) \middle| \Omega \right\rangle \tag{18}$$

with the time-ordering operator defined as

$$\mathscr{T} \mathbf{\Phi}(\underline{x}_1) \mathbf{\Phi}(\underline{x}_2) = \Theta(t_1 - t_2) \mathbf{\Phi}(\underline{x}_1) \mathbf{\Phi}(\underline{x}_2) + \Theta(t_2 - t_1) \mathbf{\Phi}(\underline{x}_2) \mathbf{\Phi}(\underline{x}_1). \tag{19}$$

(e) finally prove that Δ_+ and Δ fulfill the free Klein"=Gordon equation,

$$(\Box_1 + m^2)\Delta_+(\underline{x}_1, \underline{x}_2) = (\Box_1 + m^2)\Delta(\underline{x}_1, \underline{x}_2) = 0 \tag{20}$$

and that D_{ret} as D_{F} are Green's functions of the Klein"=Gordon operator, i.e.,

$$(\Box_1 + m^2)\mathrm{D}_{\text{ret}}(\underline{x}_1, \underline{x}_2) = (\Box_1 + m^2)\mathrm{D}_{\text{F}}(\underline{x}_1, \underline{x}_2) = -\delta^{(4)}(\underline{x}_1 - \underline{x}_2). \tag{21}$$