Institut für Theoretische Physik Goethe
Universität Frankfurt
WiSe 2025/2026
PD Dr. Hendrik van Hees

Exercise Sheet 8

1. Gauge invariance in QED

Consider the of quantum electrodynamics:

$$\mathscr{L}_{\text{QED}} = \bar{\psi}(\mathrm{i}\not{D} - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}. \tag{1}$$

with $\mathrm{D}_\mu = \partial_\mu + \mathrm{i}qA_\mu$.

(a) Show that $\mathscr{L}_{\text{QED}}$ is invariant under the gauge transformations

$$\begin{aligned} \psi(x) &\mapsto e^{-\mathrm{i}q\alpha(x)}\psi(x), \\ A_\mu(x) &\mapsto A_\mu(x) + \partial_\mu\alpha(x). \end{aligned} \tag{2}$$

(b) Show that only adding a mass term $\frac{1}{2}MA^\mu A_\mu$ for the photon breaks gauge invariance.

(c) Show that adding a free real vector field $\theta(x)$ to the theory and adding

$$\mathscr{L}_{\text{Stückel}} = \frac{1}{2}(\partial_\mu\theta)(\partial^\mu\theta) + MA^\mu\partial_\mu\theta \tag{3}$$

to the QED Lagrangian restores gauge invariance despite the mass term for A_μ, if one transforms θ in a clever way.

(d) It is commonly said that gauge symmetries reflect "a redundancy in the mathematical description of the system". Then why do we demand it to be respected in a physical theory?

2. Polarizations of the photon

(a) From the QED Lagrangian (1), derive the equations of motion for the ψ and A_μ. How can you connect the latter with the Maxwell's equations $\partial_\mu F^{\mu\nu} = j^\nu$ from classical electrodynamics?

(b) The Lorenz gauge fixing condition $\partial_\mu A^\mu = 0$ is *incomplete*, that is, we can still make another transformation

$$A_\mu(x) \mapsto A_\mu(x) + \partial_\mu\Lambda(x). \tag{4}$$

Determine the condition on Λ for this to be true.

(c) The wave function for a *free photon* satisfies the equation

$$\Box A^\mu = 0, \tag{5}$$

which has solutions

$$A^\mu = \varepsilon^\mu(\mathbf{k})e^{-ik\cdot x}, \quad k^2 = 0. \tag{6}$$

The *polarization vector* ε^μ has 4 components! How can it describe a spin-1 particle?

(d) Based on your answer to b), choose a convenient gauge parameter to show that physics is unchanged by the transformation

$$\varepsilon^\mu \mapsto \varepsilon^\mu + ak^\mu, \tag{7}$$

for some constant a. In other words, two polarization vectors differing by a multiple of k describe the same free photon. We can use this freedom to set $\varepsilon^0 \equiv 0$. Then, what happens to the Lorenz condition?