Gapless Hartree-Fock approximations for the linear σ model

Hendrik van Hees

Justus-Liebig-Universität Gießen

June 10, 2008

with

Yu. B. Ivanov, F. Riek, J. Knoll

Institut für Theoretische Physik

JUSTUS-LIEBIG-UNIVERSITÄT GIESSEN

Outline

1. Φ-derivable (2PI) approximation schemes
2. Symmetry violations in 2PI approximations
3. Solution for the linear σ model
4. Conclusions

Local and bilocal sources

- Generating functional for (disconnected) Green's functions

$$Z[J,B] = N \int \mathrm{D}\phi \exp\left[\mathrm{i}S[\phi] + \mathrm{i}\left\{J_1\phi_1\right\}_1 + \frac{\mathrm{i}}{2}\left\{B_{12}\phi_1\phi_2\right\}_{12}\right]$$

- Generating functional for connected Green's functions

$$W[J,B] = -\mathrm{i}\ln Z[J,B], \quad \frac{\delta W}{\delta J_1} = \varphi_1, \quad \frac{\delta W}{\delta B_{12}} = \frac{1}{2}\left(G_{12} + \varphi_1\varphi_2\right)$$

- Legendre transform: 2PI generating functional

$$\Gamma[\varphi, G] = W[J,B] - \left\{J_1\varphi_1\right\}_1 - \frac{1}{2}\left\{(\varphi_1\varphi_2 + \mathrm{i}G_{12})B_{12}\right\}_{12}$$

- Saddle point expansion of the path integral

$$\Gamma[\varphi, G] = S[\varphi] + \frac{\mathrm{i}}{2}\operatorname{Tr}\ln(\beta^2 G^{-1}) + \frac{\mathrm{i}}{2}\left\{D_{12}^{-1}(G_{12} - D_{12})\right\}_{12} + \Phi[\varphi, G]$$

with $\quad D_{12}^{-1} = \dfrac{\delta^2 S[\varphi]}{\delta\varphi_1\delta\varphi_2}$

Equations of Motion

- Want to find φ and G at vanishing external sources $\Rightarrow$ Equations of motion:

$$\frac{\delta\Gamma}{\delta\varphi_1} = j_1 + \{B_{12}\varphi_2\}_2 \stackrel{!}{=} 0, \quad \frac{\delta\Gamma}{\delta G_{12}} = -\frac{\mathrm{i}}{2} B_{12} \stackrel{!}{=} 0$$

- Second equation:

$$D_{12}^{-1} - {G_{12}}^{-1} = 2\mathrm{i}\frac{\delta\Phi}{\delta G_{12}} = \Sigma_{12}$$

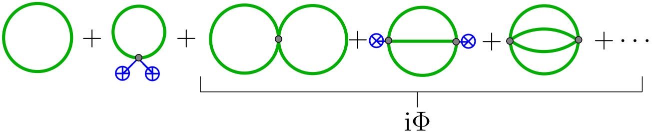
- Φ generates skeleton diagrams for self-energy
- Φ must be two-particle irreducible (2PI)
- Saddle-point expansion of the path integral: Φ diagrams ≥ 2 loops

Simple ϕ^4 model

$$\mathscr{L} = \frac{1}{2}(\partial_\mu \phi)(\partial_\mu \phi) - \frac{m}{2}\phi^2 - \frac{\lambda}{2}\phi^4, \quad S[\phi] = \{\mathscr{L}_1\}_1$$

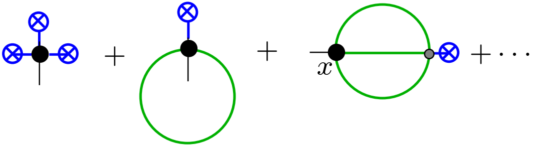
The functional:

$$\mathrm{i}\Gamma[\varphi, G] = \mathrm{i}S[\varphi] +$$



Field equation of motion:

$$\mathrm{i}(\Box + m^2)\varphi =$$



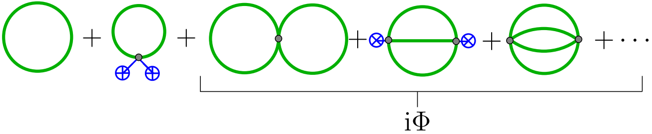
“Diagrammar”

Simple ϕ^4 model

$$\mathscr{L} = \frac{1}{2}(\partial_\mu \phi)(\partial_\mu \phi) - \frac{m}{2}\phi^2 - \frac{\lambda}{2}\phi^4, \quad S[\phi] = \{\mathscr{L}_1\}_1$$

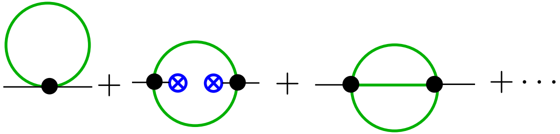
The functional:

$\mathrm{i}\Gamma[\varphi, G] = \mathrm{i}S[\varphi] +$



Self energy:

$-\mathrm{i}\Sigma_{12} =$



Why should one use the Φ functional?

- Provides a self-consistent set of equations of motion
- Approximations yield equations, which
  - lead to conserved expectation values of Noether currents
  - $\mathrm{i}\Gamma = \ln Z$ at the solution
    (a non-perturbative approximation of the partition sum)
  - allows consistent determination of thermodynamical and dynamical properties through analytic properties of Green's functions
- especially useful for description of particles and resonances with finite mass width
- only way to find self-consistent equation with these properties!

[Baym 1962]

Breaking of symmetries: The O(N)-σ model

$$\mathscr{L} = \frac{1}{2}(\partial_\mu \vec{\phi})(\partial^\mu \vec{\phi}) - \frac{m}{2}\vec{\phi}^2 - \frac{\lambda}{4N}\left(\vec{\phi}^2\right)^2$$

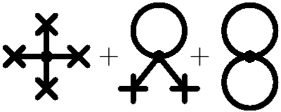
- Action symmetric under global O(N) rotations of $\vec{\phi}$
- Symmetry linear $\Rightarrow$ exact Quantum action also symmetric
- perturbative loop expansion $=$ power expansion in $\hbar \Rightarrow$ also symmetric at any finite order of pert. theory
- If symmetry spontaneously broken $(m^2 < 0)$, from this symmetry alone follows Goldstone's theorem: There are $N-1$ massless Goldstone bosons
- Long known [Baym, Grinstein 1977]: Φ-derivable approximations break the symmetry explicitly!
- Goldstone's theorem also violated

Gapless Φ-derivable approximations

- Φ-derivable approximation which fulfills Nambu-Goldstone theorem

  [Yu. B. Ivanov, F. Riek, J. Knoll 2005]

⇒ Construct "correction" ΔΦ to Φ functional such that
  - Nambu-Goldstone theorem is fulfilled in spont. broken phase
  - in symmetric phase: same EoMs in symmetric phase as original approximation
  - EoM for mean field unchanged
- for Hartree-Fock approximation

$$\Phi_{\text{gHF}} =$$  $$+ \Delta\Phi$$

$$\Delta\Phi = -\frac{\lambda}{2N}\left[N Q_{ab} Q_{ab} - (Q_{aa})^2\right]$$

$$= Q_{ab} = \int_\beta d^4k G_{ab}(k)$$

Renormalization

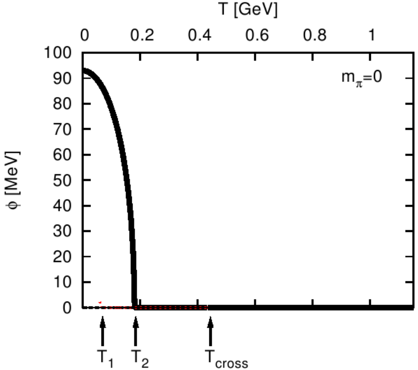
- mass-independent renormalization scheme

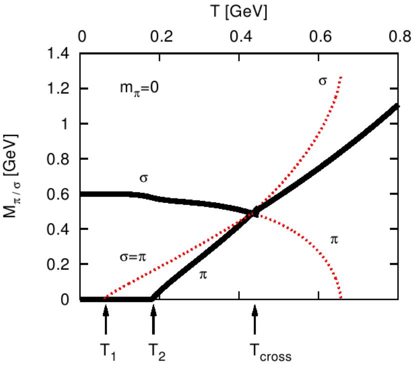
$$\Sigma_{\text{vac}}(\phi=0, m^2=\mu^2, p^2=0)=0,$$
$$\partial_{m^2}\Sigma_{\text{vac}}(\phi=0, m^2=\mu^2, p^2=0)=0,$$
$$\partial_{p^2}\Sigma_{\text{vac}}(\phi=0, m^2=\mu^2, p^2=0)=0$$

- preserves $O(N)$ symmetry
- only vacuum counter terms needed in Φ-derivable scheme [HvH,J. Knoll 2002]
- similar conditions used for effective potential

Solutions for $O(4)$ model in chiral limit

- With $\mu = 600$ MeV
- fixed physical parameters in vacuum: $m_\sigma = 600$ MeV, $f_\pi = 93$ MeV

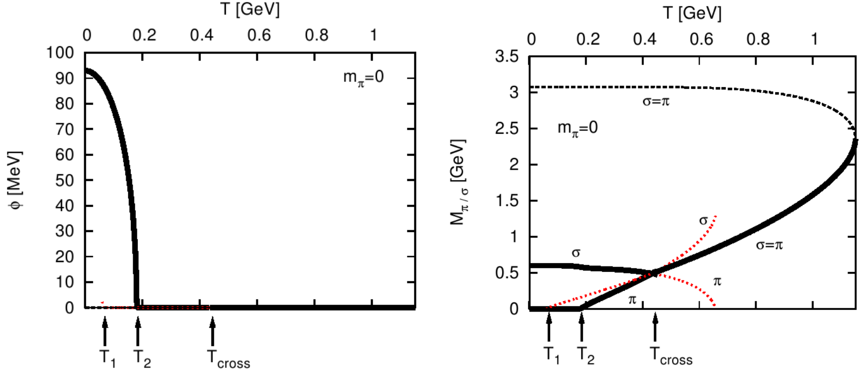




- stable and meta stable solutions
- 2nd-order phase transitions
- $m_\pi = 0$ in spont. broken phase ($\phi = 0$)

Solutions for $O(4)$ model in chiral limit

- With $\mu = 600$ MeV
- fixed physical parameters in vacuum: $m_\sigma = 600$ MeV, $f_\pi = 93$ MeV



- another high-mass metastable branch
- no solutions at $T > T_{\text{end}}$
- effective renormalized coupling becomes high!
- approximation unreliable

Conclusions

- For linear $O(N)$-σ model
  - Φ-derivable (2PI) gapless approximations
  - renormalizable with symmetry-preserving vacuum counter terms
  - renormalization-scale independent vacuum solutions
  - stability of vacuum model: $\mu > \mu_0$
  - at finite temperature: 2^{nd}-order phase transition(s)
  - various stable and meta-stable solutions
  - model breaks down at $T > T_{\text{end}}$
- remaining problems
  - at finite T: renormalization-scale dependence
  - deviation of renormalization-group β from perturbation theory at the same order [E. Braaten, E. Petitgirard 2005; C. Destri and A. Sartirana 2005]
  - reason: subtraction of "hidden divergence" of the coupling constant resummed only in one channel
  - only partial resummation $\Rightarrow$ breaking of crossing symmetry at orders higher than expansion parameter like λ, $\hbar$
- Feasibility of gapless Φ-derivable approximations at higher orders including scattering (sunset diagrams)?