The Cosmological Constant

One of the problems in modern physics

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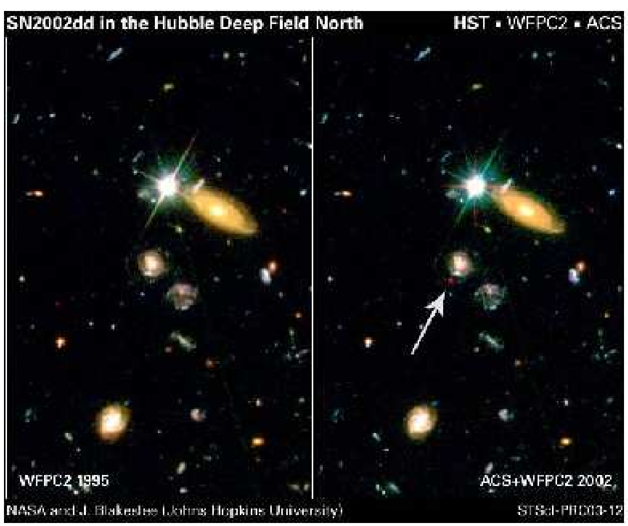

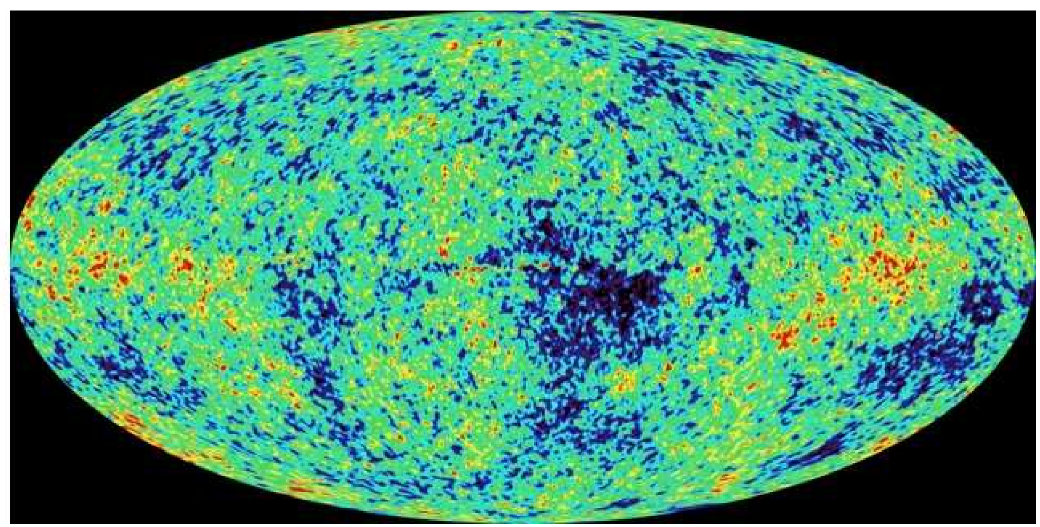

Contents

Space-time geometry

As I was going up the stair,
I met a man who wasn't there.
He wasn't there again today,
I wish, I wish he'd stay away

Hughes Mearns (Cited from [7]).

- In a sense it was always there. Principle of general relativity (invariance under local GL$(4, \mathbb{R}^4)$) and equivalence principle (gravity locally equal to accelerated frame of reference)

$\Rightarrow$ Gravity $=$ curvature of space time, which is a pseudo-Riemannian manifold

- Look for GL$(4, \mathbb{R}^4)$-invariant Lagrangian of as a function of $g_{\mu\nu}$ and $g_{\mu\nu,\rho}$.
- Such a Lagrangian doesn't exist, but from the curvature tensor

$$\Gamma_{\mu\nu\rho} = (g_{\mu\nu,\rho} + g_{\mu\rho,\nu} - g_{\nu\rho,\mu}), \quad \Gamma^{\rho}{}_{\mu\nu} = g^{\rho\alpha}\Gamma_{\alpha\mu\nu},$$
$$R^{\rho}{}_{\mu\nu\sigma} = \Gamma^{\rho}{}_{\mu\sigma,\nu} - \Gamma^{\rho}{}_{\mu\nu,\sigma} + \Gamma^{\rho}{}_{\alpha\nu}\Gamma^{\alpha}{}_{\mu\sigma} - \Gamma^{\rho}{}_{\alpha\sigma}\Gamma^{\alpha}{}_{\mu\nu}$$

we get the Ricci tensor and the curvature scalar

$$R_{\mu\nu} = R^{\rho}{}_{\mu\nu\rho}, \quad R = R_{\mu\nu}g^{\mu\nu}.$$

Einstein-Hilbert action

- R contains derivatives of $g_{\mu\nu}$ to second order, but only linear with coefficients independent of derivatives
- The curvature scalar R is the only scalar of the pseudometric with these properties except a constant! $\Rightarrow$ Most general action

$$S[g, \text{matter fields}] = \frac{1}{2\kappa} \int \mathrm{d}^4x \sqrt{-g}(R+\Lambda)+S_{\text{matter}} \text{ with } \kappa = \frac{8\pi G}{c^2} = 1.865{\cdot}10^{-27}\,\frac{\text{cm}}{\text{g}}$$

- Variation leads to Einstein's Equations (Einstein, Hilbert 1915)

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = -\kappa T_{\mu\nu}$$

- $T_{\mu\nu}$ Belinfante energy-momentum tensor of the matter contribution to the action:

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta S_{\text{matter}}}{\delta g^{\mu\nu}}$$

- Symmetric and gauge invariant for gauge fields!

Cosmology

- Cosmological principle: There exists a fundamental frame of reference where the time slice ("3-space" of an observer defining this frame) which is homogeneous and isotropic. Determines the metric uniquely (up to coordinate transformations):

$$ds^2 = dt^2 - a^2(t)\left[\frac{\mathrm{d}r^2}{1-kr^2} + r^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2)\right]$$

- $k \in \{-1, 0, 1\}$, corresponding to a hyperbolical (open, infinite), a flat (open, infinite), or a spherical (closed, without boundary) 3-space
- Einstein's equations $\Rightarrow$ specialize to Friedmann's equation

$$\frac{3\ddot{a}}{a} = \Lambda - \frac{\kappa}{2}(\rho_M + 3p_M), \qquad \left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} - \frac{\Lambda}{3} = \frac{\kappa}{3}\rho_M$$

  where ρ_M and p_M are the density and pressure of matter (modelled as an ideal fluid and/or radiation)
- radiation: $p_r = \rho_r/3$, "dust" $p_m = 0$
- Cosmological constant can be positive: Then it leads to a "repulsion", otherwise to "attraction" like ordinary matter ($\rho, p > 0$).

Problems with matter models

- Fundamentals of local quantum field theory: "Quantization" of the free fields yields infinite "vacuum energy"
- Each bosonic field has positive vacuum energy density
- each fermionic field has negative vacuum energy density
- Regularized with a cut-off k_{max}:

$$\langle 0 \, | \boldsymbol{\rho} | \, 0 \rangle = \pm \int_0^{k_{\text{max}}} \frac{4\pi k^2 \mathrm{d}k}{(2\pi)^3} \frac{1}{2} \sqrt{k^2 + m^2} = \frac{1}{16\pi^2} [k_{\text{max}}^4 + m^2 k_{\text{max}}^2 + O(m^4)]$$

- No problem for elementary particle theory (without gravitation): Adjust the vacuum energy to 0 due to the assumption of Poincaré invariance
- only differences of total energy observable
- General relativity: contribution to the Cosmological constant
- Setting $k_{\text{max}} = M_{\text{Planck}} = (8\pi G)^{-1/2}$:

$$\langle 0 \, | \boldsymbol{\rho} | \, 0 \rangle \approx 2 \cdot 10^{71} \text{GeV}^4$$

Fine-tuning problem

- Observations: Measurements of density parameter together with flatness of the universe from inflation

$$\Omega_{\text{tot}} = \Omega_M + \Omega_\Lambda \approx 1, \quad \Omega := \frac{\rho}{\rho_{\text{crit}}}, \quad \rho_{\text{crit}} = (3.0 \cdot 10^{-3}\text{eV})^4 h^2$$

- Magnitude-redshift relation for distant type Ia supernovae ($\Rightarrow$ standard candles)
- Measurements from WMAP + HST:

$$\Omega_{\text{tot}} = 1.02 \pm 0.02, \quad \Omega_\Lambda = 0.73 \pm 0.04, \quad h = 0.71 + 0.03 - 0.04 \Rightarrow \Lambda = 4 \cdot 10^{-47}\text{GeV}^4$$

- Vacuum energy of field degree of freedom by a factor of 10^{118} (!!!) too large
- fine-tuning with 118 digits precision needed
- in standard model of electroweak interactions: Higgs fields

$$V = V_0 - \mu^2 \phi^\dagger \phi + g(\phi^\dagger \phi)^2, \quad \left\langle 0 \middle| \boldsymbol{\rho}_{\text{Higgs}} \middle| 0 \right\rangle = V_{\text{min}} = V_0 - \mu^4/(4g)$$

- What's the "right" value for V_0? Again fine-tuning for the observed value of Λ!
- Hubble expansion accelerated today. Why?

The problem from a general perspective

- Looking for field equations satisfying translational covariance $\Rightarrow$ all fields must be constant and satisfy

$$\partial\mathscr{L}/\partial\psi_i = 0 \ (i \in \{1,\dots,N\}), \quad \partial\mathscr{L}/\partial g_{\mu\nu} = 0$$

- in GRT $\mathscr{L}$ is GL(4)-symmetric $\Rightarrow$ for constant fields this means the theory must be invariant under

$$g_{\mu\nu} \to A^\rho{}_\mu A^\sigma{}_\nu g_{\rho\sigma}; \quad \psi_i \to D_{ij}(A)\psi_j, \quad \mathscr{L} \to \mathscr{L}\cdot\det A \text{ with } A^\rho{}_\mu, D_{ij} = \text{const.}$$

- If $\partial\mathscr{L}/\partial\psi_i$ is fulfilled for constant ψ_i then $\mathscr{L}$ is invariant under the global GL(4)-transformation of g

$$\mathscr{L} = c\sqrt{-g} \text{ with } c = \text{const.}$$

- $\partial\mathscr{L}/\partial g_{\mu\nu} = 0$ only if by fine-tuning $c = 0$
- Ways out: symmetry principles that prevent the appearence of a cosmological constant or “cosmo-dynamical” solutions that drive the cosmological constant necessarily to 0 due to the dynamical evolution of the universe.

Supersymmetry

- Supersymmetry generators Q_α

$$\left[Q_\alpha, Q^\dagger_\beta\right]_+ = \mathbf{P}^\mu [\sigma_\mu]_{\alpha\beta} \text{ with } \sigma_0 = 0,\ \sigma_j = \text{Pauli matrices}$$

- Unbroken SUSY:

$$Q_\alpha \left|0\right\rangle = Q^\dagger_\alpha \left|0\right\rangle = 0 \Rightarrow \left\langle 0 \left| \mathbf{P}^\mu \right| 0 \right\rangle = 0.$$

  Unbroken SUSY implies vanishing energy and momentum of the vacuum $\Rightarrow$ cosmological constant vanishes without fine-tuning!

- Field theoretical reason: For each boson there must be a fermionic partner and vice versa $\Rightarrow$ Vacuum energies cancel exactly
- Quantum corrections: boson and fermion loops exactly cancel each other's contribution to the vacuum energy
- But SUSY is broken $\Rightarrow$ positive definite vacuum energy
- Also extension to a supersymmetric supergravity theory does not really help

Dynamical solutions?

- There seems to be no symmetry principle which prevents the necessity of fine-tuning
- Need to explain only why the cosmological constant is small now
- Idea: Scalar field with

$$\Box\phi \propto T_\mu{}^\mu,$$

  where $T^{\mu\nu}$ is the *total energy momentum tensor* including $\Lambda g^{\mu\nu}/(8\pi G)$
- Further assumption $T^\mu{}_\mu$ vanishes for $\phi = \phi_0$
- Then ϕ evolves to its equilibrium value ϕ_0 and Einstein's equations have the flat-space solution
- make ϕ weakly coupled, so that it is unobservable; then ϕ_0 very large
- Suppose ϕ has a small mass m_ϕ. For small momenta of range $|p| \ll m_\phi$ we get an effective theory for the massless fields (e.g., gravitational and electromagnetical)
- To cancel the vacuum energies of these fields, such that $\rho_{\text{vac}} < 10^{-48}\text{GeV}^4$, we must have $m_\phi < 10^{-12}\text{GeV}$
- Does such a model exist?

Weinberg's no-go theorem I

- Look again for constant field solutions of Einstein's + matter field equations

$$\frac{\partial \mathscr{L}}{\partial g_{\mu\nu}} = 0, \quad \frac{\partial \mathscr{L}}{\partial \psi_n} = 0$$

- To make Λ vanishing we must satisfy $g_{\mu\nu}\partial\mathscr{L}/\partial g_{\mu\nu} = 0$
- The natural way to do so for constant fields is to demand

$$g_{\mu\nu}\frac{\partial \mathscr{L}}{\partial g_{\mu\nu}} = -\sum_n \frac{\partial \mathscr{L}}{\partial \psi_n} f_n(\psi)$$

- For fulfilled equations for ψ_n the Lagrangian must be of the form $\mathscr{L} = c(\psi)\sqrt{-g}$. Symmetry of the Lagrangian:

$$\delta g_{\mu\nu} = -2\delta\epsilon g_{\mu\nu}, \quad \delta\psi_n = \delta\epsilon f_n(\psi)$$

Weinberg's no-go theorem II

- can always redefine the fields ψ_n such that one has $N-1$ fields σ_a and one scalar ϕ such that the symmetry transformation reads

$$\delta g_{\mu\nu} = -2\delta\epsilon g_{\mu\nu}, \quad \delta\sigma_a = 0, \quad \delta\phi = \epsilon$$

- For constant fields $\mathscr{L} = \mathscr{L}[\exp(2\phi)g_{\mu\nu}, \sigma]$:

$$\mathscr{L} = \exp(4\phi)\sqrt{-g}\mathscr{L}_0(\sigma) \;\Rightarrow\; \frac{\partial\mathscr{L}}{\partial\phi} = -T^{\mu}{}_{\mu}\sqrt{-g}, \quad T^{\mu\nu} = -g^{\mu\nu}\exp(4\phi)\mathscr{L}_0(\sigma).$$

- All would be nice, if there was a stationary point of $\mathscr{L}$ for some ϕ, but $\exp(\phi)$ has no stationary points!

Quintessence

- Try to solve cosmological constant problem with non-constant $g_{\mu\nu}$
- Uniform scalar field $\phi(t)$ (see lectures about inflation)

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0, \quad H = \sqrt{\frac{3}{8\pi G}(\rho_\phi + \rho_M)}$$

- energy densities of scalar field and matter+radiation:

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi), \quad \dot{\rho}_M = -3H(\rho_M + p_M)$$

- Potential V: ϕ approaches the value with $V'(\phi_0) = 0$, near this value ϕ changes slowly with time
- ρ_M decreasing with time ($H > 0$). Then slowly varying Hubble parameter $H \approx \sqrt{8\pi G V(\phi)/3}$ (exponential expansion)
- Problem: Why should $V(\phi)$ be small (zero) where $V'(\phi) = 0$?
- Coincidence problem: Need very carefully set initial conditions

Tracker solutions

- Special choice of V (tracker solutions):

$$V(\phi) = M^{4+\alpha}\phi^{-\alpha}, \quad \alpha > 0, \quad M = \text{const.}$$

- ϕ starts with value $\ll M_{\text{Planck}}$ and $\dot{\phi}^2 \ll \rho_m$ then $\phi(t) \sim t^{2/(2-\alpha)}$, $\rho_\phi \sim t^{-2\alpha/(2+\alpha)}$, $\rho_M \sim t^{-2}$ decreasing faster $\Rightarrow$ good for cosmological nucleosynthesis, because ρ_M dominates at temperatures of 10^9K - $10^{10}K$
- Later ρ_ϕ dominates over ρ_M, $\rho_\phi \sim t^{-2/(4+\alpha)}$, $H \sim \sqrt{V(\phi)} \sim t^{-\alpha/(4+\alpha)}$
- ρ_ϕ-dominance today! ρ_M and ρ_ϕ both contribute to cosmic expansion rate
- Good thing: no fine-tuning for cross-over from ρ_M-domination to ρ_ϕ-domination
- Doesn't solve the Λ-problem, because why shouldn't there be an additional constant $\sim m^4_{\text{Planck}}$ to $V(\phi)$, i.e., no naturalness of above choice $\Rightarrow$ fine-tuning necessary again
- Even if $V(\phi)$ chosen as above, need fine-tuning for M such that $\rho_\phi \approx \rho_M$ close to the present critical density ρ_{c0}:

$$M^{4+\alpha} \approx \frac{\rho_{c0}}{(8\pi G)^{\alpha/2}} \approx \frac{H_0^2}{(8\pi G)^{-1-\alpha}} \qquad (1)$$

K-Essence

- Idea: Modify the kinetic energy of the scalar field

$$\mathscr{L} = -\frac{1}{6}R + p(\phi, X), \quad X = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi), \quad p = K(\phi)\tilde{p}(X)$$

- $p =$ pressure, energy density

$$\rho_\phi = K(\phi)[2X\partial_X \tilde{p}(X) - \tilde{p}(X)]$$

- Depending on $K(\phi)$ and $\tilde{p}(X)$ we can have trackers, i.e., k-essence mimics the e.o.s. of the matter and/or radiaton content of the universe
- or attractors: k-essence is driven to a e.o.s. different from matter or radiation
- Insensitive to initial conditions
- Provides negative pressure after some time of matter domination ($w = p/\rho \sim -1$) and today, acts like an effective positive cosmological constant
- Only a tracking solution if radiation dominated epoch
- Further behavior after overtaking matter energy density depends on the details of the model: It can be $w < -1/3$ (forever accelerating universe) or $-1/3 < w \leq 0$ (decelerating or dust-like)

Summary

- Existence of cosmological constant from geomtry of space time
- Fields, describing matter+radiation: Contribute to (effective) Λ
- Quantization of fields $\Rightarrow$ vacuum energy $\Rightarrow$ "1st fine-tuning problem"
- Data: Flat universe $\Omega_{\text{tot}} \approx 1 \Rightarrow$ Inflation
- Weinberg's no-go theorem: No "natural" model with $g_{\mu\nu} = \eta_{\mu\nu}$
- Quintessence, tracker solutions $\Rightarrow$ Always "dynamical universe"
- 2nd fine-tuning problem: make Ω_M and Ω_Λ and $\Omega_{\text{tot}} = 1$ as observed and accelerating expansion
- K-essence: Looks like the only so far found solution to both problems
- *Is that true? What about quantization?*

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