Comment about “Didactical formulation of the Ampère law”

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Abstract. In this comment we demonstrate that the example of a DC conducting wire of finite length as and example for the application of the Biot-Savart Law is flawed due to the non-conservation of electric charge for such an unphysical setting. The implications drawn in [1] on the restrictions of Ampère’s Circuital Law in integral form are unnecessary as long as only physically realizable situations obeying the charge-conservation law are considered.

1. Foundations

The following considerations can be found in any textbook on electromagnetics, e.g., in [2, 3, 4]. The foundation of electrodynamics from a modern point of view is that it provides the paradigm of a (classical) relativistic local field theory with a "massless" vector field, which necessarily must be described as a gauge field from the point of view of representation theory of the Poincaré group. With this paradigm in mind, one should teach electromagnetics emphasizing these underlying principles. Particularly the theory is most clearly set up with the local Maxwell equations in vacuo. In this manuscript we restrict ourselves to the most simple application of magnetostatics, i.e., the calculation of the magnetic field, $\vec{B}$, for a given electric current-density distribution, $\vec{j}$. Assuming stationarity, in this case the Maxwell equations separate into equations for the electric and magnetic field components separately, and we consider only the magnetic field.

In the vacuum, using Heaviside-Lorentz units, the magnetostatic equations in vacuo read

$$\vec{\nabla} \cdot \vec{B} = 0, \tag{1}$$

and

$$\vec{\nabla} \times \vec{B} = \frac{\vec{j}}{c}. \tag{2}$$

The first Law (Gauss's Law for the magnetic field) states that there are no magnetic monopole charges, and the second Law (Ampère's Law) states that in the static limit the electric current density provides the vortex source of the magnetic field.

A very important point, related to gauge invariance, is that from (2) it immediately follows that the current density must fulfill the integrability constraint

$$\vec{\nabla} \cdot \vec{j} = 0, \tag{3}$$

which is the stationary version of the **local charge-conservation** law. That this follows without the consideration of the detailed equations of motion for the charged particles making up the current density is due to the gauge-invariance constraint and of utmost importance for the physicality of the solutions for the electromagnetic field.

The solution of the coupled set of field equations (1) and (2), **given the constraint (3)**, starts with the introduction of a vector potential for the magnetic field, making use of Helmholtz's fundamental theorem of vector calculus:

$$\vec{B} = \vec{\nabla} \times \vec{A}. \tag{4}$$

The vector potential is determined only up to an additive gradient field, which reflects gauge invariance for the static magnetic field. This "gauge freedom" can be used to simplify the task to solve for (2). With (4) the Ampère Law reads

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \frac{\vec{j}}{c}. \tag{5}$$

In Cartesian coordinates we can write

$$\vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \Delta \vec{A} = \frac{\vec{j}}{c}. \tag{6}$$

Choosing the Coulomb-gauge constraint,
$$\vec{\nabla} \cdot \vec{A} = 0, \tag{7}$$
as an auxiliary condition yields the Poisson equation
$$\Delta \vec{A} = -\frac{\vec{j}}{c}. \tag{8}$$
The solution in terms of the Green's function for the Laplace operator on $\mathbb{R}^3$ is immediately clear from the analogous equation for the scalar potential in electrostatics:
$$\vec{A}(\vec{x}) = \frac{1}{4\pi c} \int_{\mathbb{R}^3} \mathrm{d}^3 \vec{x}' \frac{\vec{j}(\vec{x}')}{|\vec{x} - \vec{x}'|}. \tag{9}$$
This is a valid solution **if and only if the Coulomb-gauge condition** (7) is fulfilled. Differentiation of (9) yields
$$\begin{aligned}\vec{\nabla} \cdot \vec{A}(\vec{x}) &= \frac{1}{4\pi c} \int_{\mathbb{R}^3} \mathrm{d}^3 \vec{x}' \, \vec{j}(\vec{x}') \vec{\nabla} \left(\frac{1}{|\vec{x} - \vec{x}'|} \right) \\ &= -\frac{1}{4\pi c} \int_{\mathbb{R}^3} \mathrm{d}^3 \vec{x}' \, \vec{j}(\vec{x}') \vec{\nabla}' \left(\frac{1}{|\vec{x} - \vec{x}'|} \right). \end{aligned} \tag{10}$$
and an integration by parts, assuming a sufficiently quickly vanishing current at infinity, leads to
$$\vec{\nabla} \cdot \vec{A} = \frac{1}{4\pi c} \int_{\mathbb{R}^3} \mathrm{d}^3 \vec{x}' \frac{\vec{\nabla}' \cdot \vec{j}(\vec{x}')}{|\vec{x} - \vec{x}'|} \stackrel{(3)}{=} 0 \tag{11}$$
which shows that (10) is a valid solution for the vector potential if and only if the charge-conservation condition (3) is fulfilled.

Taking the curl of (9) yields the **Biot-Savart Law**
$$\vec{B}(\vec{x}) = -\frac{1}{4\pi c} \int_{\mathbb{R}^3} \mathrm{d}^2 \vec{x}' \frac{\vec{x} - \vec{x}'}{|\vec{x} - \vec{x}'|^3} \times \vec{j}(\vec{x}'). \tag{12}$$
From the derivation it is clear that this is a solution of the magnetostatic problem if and only if (3) is fulfilled. It is clear that for a field given by (12) always fulfills (1), even if (3) is violated, because it can be written as the curl of a vector potential defined by (9). In this case, however, (2) is not fulfilled anymore, because then the Coulomb-gauge constraint (7) is violated and thus the solution of (8) is not equivalent to the correct equation (5).

This is clearly demonstrated on hand of the example of a finite current-conducting wire brought up in [1], which clearly violates (3) and is unphysical since it cannot fulfill the electric part of the equations with all physical boundary conditions, except the trivial electrostatic case, leading to $\vec{j} = 0$ and $\vec{E} = 0$ in the interior of the conductor.

2. Example of the finite wire

In [1] a straight wire along the z axis of length L ($-L/2 < z < L/2$) is considered. Assuming a constant total current I is running in this finite piece of wire, the current density is given by
$$\vec{j}(\vec{x}) = I \delta(x) \delta(y) \Theta \left(\frac{L}{2} - |z| \right) \vec{e}_z, \tag{13}$$

where δ denotes the Dirac-δ distribution and Θ the Heaviside-unit-step function. It is clear from the very beginning that this violates (3), because by a straight-forward differentiation one finds

$$\vec{\nabla} \cdot \vec{j}(\vec{x}) = I\delta(x)\delta(y) \left[\delta\left(z+\frac{L}{2}\right) - \delta\left(z+\frac{L}{2}\right)\right] \neq 0. \tag{14}$$

Using nevertheless (12) gives after the trivial integration over x and y

$$\begin{aligned}\vec{B}(\vec{x}) &= +\frac{I}{4\pi c}\vec{e}_z \times \vec{x} \int_{-L/2}^{L/2} \mathrm{d}z' \frac{1}{[x^2+y^2+(z-z')^2]^3} \\ &= \frac{I}{4\pi c R}\vec{e}_\vartheta \left[\frac{L-2z}{\sqrt{4R^2+(2z-L)^2}} + \frac{L+2z}{\sqrt{4R^2+(2z+L)^2}}\right],\end{aligned} \tag{15}$$

where we have introduced the standard cylinder coordinates (R, ϑ, z). Evaluating the curl of (15) reveals that this result does not solve the Ampère Law (2) as expected from the violation of the charge-conservation constraint (3). Taking the curl of (15) by performing the derivatives, one indeed gets

$$\vec{\nabla} \times \vec{B} = \frac{J}{\pi c}\left\{\frac{2R\vec{e}_R + (2z-L)\vec{e}_z}{[4R^2+(2z-L)^2]^{3/2}} - \frac{2R\vec{e}_R - (2z+L)\vec{e}_z}{[4R^2+(2z+L)^2]^{3/2}}\right\} \neq \frac{1}{c}\vec{j}. \tag{16}$$

Strictly speaking this evaluation is valid only at the non-singular points of $\vec{B}$, but that's only the z axis, and for $R \neq 0$ we should obtain 0 for (2) to hold, and that is not the case! Thus, indeed the Biot-Savart Law (12) is not a solution of (2). Of course (15) fulfills (1), as expected from the above derivation of the Biot-Savart Law. A straight-forward evaluation confirms

$$\vec{\nabla} \cdot \vec{B} = 0. \tag{17}$$

The same result can also be achieved by evaluating the vector potential according to (9):

$$\begin{aligned}\vec{A}(\vec{x}) &= \frac{I\vec{e}_z}{4\pi c}\int_{-L/2}^{L/2} \mathrm{d}z' \frac{1}{\sqrt{R^2+(z-z')^2}} \\ &= \frac{I\vec{e}_z}{4\pi c}\ln\left(\frac{2z+L+\sqrt{4R^2+(2z+L)^2}}{2z-L+\sqrt{4R^2+(2z-L)^2}}\right).\end{aligned} \tag{18}$$

Indeed, taking the curl leads back to (15). Here the flaw according to the non-conservation of electric charge can be seen from the fact that

$$\vec{\nabla} \cdot \vec{A} = \frac{I}{2\pi c}\left[\frac{1}{\sqrt{4R^2+(2z+L)^2}} - \frac{1}{\sqrt{4R^2+(2z-L)^2}}\right] \neq 0. \tag{19}$$

As is clear from the derivation above, (9) provides a correct solution of the magnetostatic Maxwell equations in terms of a vector potential if and only if the Coulomb-gauge constraint (7) is fulfilled. Of course, (18) is a solution of (8) for the current density (13), but since (7) is not fulfilled, that is *not* the same as $\vec{\nabla} \times (\vec{\nabla} \times \vec{A})$ entering the correct equation (5), as we have already shown in (16).

Another way to accommodate the idea of a wire of finite length carrying a constant current with classical electrodynamics has been provided in [5]. There the violation of current conservation has been fixed by introducing time-varying point charges at the end of the wire such that the full equation of continuity,

$$\partial_t \rho + \vec{\nabla} \cdot \vec{j} = 0, \tag{20}$$

is fulfilled by setting

$$\rho = (Q_0 - It)\delta(x)\delta(y) \left[\delta\left(z + \frac{L}{2}\right) - \delta\left(z + \frac{L}{2}\right)\right], \tag{21}$$

where Q_0 is an arbitrary integration constant. Then, of course, the problem is not static anymore, and one has to solve for the full time-dependent problem, using the retarded potentials in Lorenz gauge or, equivalently, Jefimenko's equations. As it turns out the exact magnetic field of this charge-current configuration is then indeed given by (15)‡, but of course the electric field is necessarily time-dependent, and thus the electromagnetic field does not fulfill the static Ampère Law anymore but the full Maxwell-Ampére Law, including the "displacement current", i.e.,

$$\vec{\nabla} \times \vec{B} - \frac{1}{c}\partial_t \vec{E} = \frac{1}{c}\vec{j}. \tag{22}$$

In any case the magnetic field (15) cannot be the solution of an entirely static situation due to the violation of charge conservation, as explained above.

3. The limit of an infinitely long wire

Contrary to the finite wire the simple well-known example of a wire of infinite length is a valid idealization of physically realizable processes since from (14) it immediately follows that the charge-conservation constraint (3) is fulfilled. Indeed, in this limit (15) becomes

$$\vec{B}(\vec{x}) = \frac{I}{2\pi c R}\vec{e}_\vartheta, \tag{23}$$

which is more easily found by applying the circuital law in integral form using the symmetries of the problem, which directly leads to the ansatz $\vec{B}(\vec{x}) = B_\vartheta(R)\vec{e}_\vartheta$. Using this expression in the integral form of (2) immediately results in (23).

Indeed, it is easy to see that (23) has a vanishing curl everywhere except along the z axis, where the field is singular due to the singular current-density distribution. A careful analysis, using the convolution with arbitrary test functions shows that (23) indeed fulfills (2) with the initially given current density (13) which in this case simplifies to

$$\vec{j}(\vec{x}) = I\delta(x)\delta(y)\vec{e}_z. \tag{24}$$

‡ Note that in [5] the usual Gaussian units have been used, while in this paper I use the rationalized Heaviside-Lorentz units.

The evaluation of the vector potential with (9) leads to a diverging integral. Indeed for $L \to \infty$ (18) becomes

$$\vec{A}(\vec{x}) = \frac{I\vec{e}_z}{2\pi c} \left[\ln\left(\frac{L}{R}\right) + \mathcal{O}(L^{-2})\right]. \tag{25}$$

Of course, a vector potential for (23) exists. It can be found by adding the gradient field

$$\vec{A}_2(\vec{x}) = \frac{I\vec{e}_z}{2\pi c} \ln\left(\frac{R_0}{L}\right) = \vec{\nabla}\left[\frac{Iz}{2\pi c} \ln\left(\frac{R_0}{L}\right)\right] \tag{26}$$

to (25), leading to

$$\vec{A}'(\vec{x}) = \frac{I\vec{e}_z}{2\pi c} \ln\left(\frac{R_0}{R}\right) \tag{27}$$

with an arbitrary constant R_0, which has no physical meaning since changing it amounts to a pure gauge transformation, which is of no effect for the magnetic field. Taking the curl of (27) indeed leads back to (23).

4. Conclusions

The above considerations show that the chosen example in [1] is misleading in that it uses the Biot-Savart Law in a situation, where it is not applicable due to the lack of charge conservation in the ansatz for the stationary current. This is a common mistake made by students in introductory courses on electromagnetism and should be avoided by stressing the importance of performing all the consistency checks, i.e., by checking that the Biot-Savart Law really leads back to the given current density when taking the curl of the $\vec{B}$ field evaluated from it. Students should also learn to always check whether the given current density fulfills the charge-conservation constraint (3), because otherwise the Ampère Law can never be fulfilled, and the Biot-Savart Law cannot lead to a proper solution of the magnetostatic Maxwell equations.

Also the necessity to revise Ampère's Circuital Law for some didactical reasons is unjustified, since it is generally valid in local form and thus, due to Stokes's integral theorem also in integral form for any surface F with a boundary ∂F

$$\int_{\partial F} \mathrm{d}\vec{x} \cdot \vec{B} = \int_F \mathrm{d}^2\vec{F} \cdot (\vec{\nabla} \times \vec{B}) = \frac{1}{c} \int_F \mathrm{d}^2\vec{F} \cdot \vec{j}. \tag{28}$$

Of course, again it is only a consistent relation, if the charge-conservation constraint (3) is fulfilled.

A necessity of a reformulation of Ampère's Circuital Law would imply nothing less than the failure of Maxwell's equations for the special case of stationary sources and fields. Given the tremendous success of classical and quantum electrodynamics in describing electromagnetic phenomena, the necessity for a revision of the Maxwell theory is pretty unlikely.

Finally we note that the above flaw is not due to the idealized situation of a thin wire (treated as a wire with vanishing radial extension) but due to the impossibility to give a steady current fulfilling the charge-conservation constraint for such a non-closed finite wire.

The integral form of Ampère's Law,

$$\frac{1}{c}\int_F \mathrm{d}^2\vec{F}\cdot\vec{j} = \int_{\partial F} \mathrm{d}\vec{x}\cdot\vec{B} \tag{29}$$

holds for all (proper) surfaces and boundaries due to Stokes's integral theorem. The suggested constraint on the boundaries in [1], is always fulfilled for any surface due to topological reasons, if the charge-conservation constraint (3) is fulfilled: Indeed, for a finite current distribution in some arbitrarily shaped wires it implies that the wires must form closed loops in order to fulfill it. Then the constraint on the surfaces given in the paper is always fulfilled: Any surface bounded by a closed loop that encircles one wire must contain one point of the wire, implying that the integration of the current density over any surface leads to the same result, namely the total current running through that wire. The same holds true for the example of an infinitely long wire. So there is, of course, no constraint to be made on the surfaces and their boundaries in the integral form of Ampére's Law, provided the charge-conservation constraint is fulfilled.

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