# Renormalization of self-consistent Schwinger-Dyson equations at finite temperature[1]

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**Abstract**

We show that Dyson resummation schemes based on Baym's Φ-derivable approximations can be renormalized with counter term structures solely defined on the vacuum level. First applications to the self-consistent solution of the sunset self-energy in $\phi^4$-theory are presented.

For the description of physical systems with strong interactions generally perturbative methods are insufficient. Rather on the basis of effective field theories non-perturbative methods such as partial resummation schemes have to be applied. Thus one works in terms of non-perturbative "dressed" propagators. One of the prominent self-consistent schemes is the so called Φ-derivable approximation [1, 2, 3] or effective action formalism [4]. There the self-energy of the dressed propagator $G$ is generated from a functional Φ by means of a variational principle

$$\Sigma_{12} = 2\mathrm{i}\frac{\delta\Phi[G]}{\delta G_{12}}, \tag{1}$$

where Φ is given by a truncated set of closed 2-particle irreducible (2PI) diagrams which cannot be split by cutting two lines. Hence Φ is the generating functional for *skeleton diagrams* and thus avoids double counting problems. It was shown by Baym [5] that for so defined approximations the expectation values of Noether charges are exactly conserved, especially those which arise from space-time symmetries (i.e., energy, momentum, angular momentum), while Ward Takahashi identities of external symmetries and even crossing symmetry for the self-energy and higher vertex functions are violated [6].

For long time the question could not be settled, to which extent self-consistent approximations of this kind can be renormalized by *temperature independent counter terms*. The problem is to isolate the divergent vacuum structure which is inherent and mostly hidden in the self-consistent self-energy due to the implied partial resummation. A further complication is that these hidden sub-divergences can be overlapping. Therefore it was necessary to analyze self-consistent partial Dyson resummation schemes defined by a set of basic generating self-energy skeleton diagrams with dressed propagators. Here skeleton diagrams are void of any self-energy insertion in any of its lines. This is guaranteed by the above mentioned 2PI property of the closed diagrams defining $\Phi[G]$.

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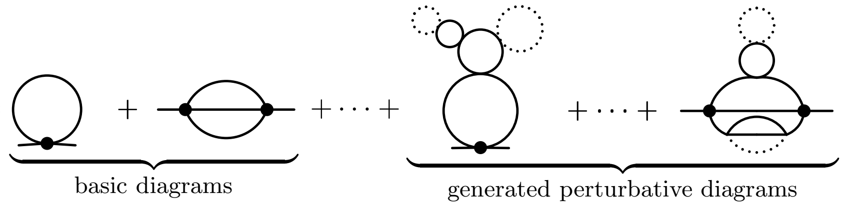
In terms of perturbative diagrams this leads to an infinite iterative insertion of all basic 1PI-diagrams. The sum of all these perturbative diagrams defines the self-consistent self-energy which determines the dressed propagator. Typical 1PI diagrams defining the self-consistent vacuum self-energy and the temperature or matter dependent pieces in $\phi^4$ theory up to the self-consistent sunset diagram are the following

$-\mathrm{i}\Sigma =$ 

(2)

Here the full lines specify the self-consistent vacuum propagators, while the dotted lines denote its temperature dependent pieces given by the KMS condition, i.e., which contain the Bose-Einstein factors. The basic diagrams define the self-consistent vacuum self-energy, which needs to be renormalized. At finite temperature the generated diagrams appear, resulting from iterative insertions of basic type diagrams, replacing one or more lines by dotted lines and omitting all diagrams which contain vacuum self energy insertions (perturbation theory starting from the self consistent vacuum level). Also in these $T$-dependent diagrams *all* subdiagrams entirely built up by vacuum propagators, which have at most four external lines, are divergent and have to be renormalized. These structures, however, appear in a nested and overlapping way such that first the innermost subdiagrams have to be renormalized through counter-terms given by the reduced diagrams where the divergent sub-pieces are contracted to a point. The so obtained reduced diagrams themselves are to be subjected to the same procedure. This iterative process is formalized as the BPHZ-renormalization scheme. For the self-consistent scheme under consideration the key issue is to find a compact iteration procedure that generates all the required counter-terms at once.

Within the real-time formalism [7, 8] we could show for the first time in general terms that such a strategy is possible [9]. In order to isolate the divergent vacuum subdiagrams from the temperature dependent remainder the use of the real-time formulation and the BPHZ-prescription of renormalization theory is crucial: By a systematic expansion around the self-consistent vacuum solution we split the self-energy and the propagator in three parts

$$\Sigma = \Sigma^{(\mathrm{vac})} + \Sigma^{(0)} + \Sigma^{(r)},\ G = G^{(\mathrm{vac})} + G^{(\mathrm{vac})}\Sigma^{(0)}G^{(\mathrm{vac})} + G^{(r)}. \tag{3}$$

Here the vacuum parts are the renormalized self-consistent *diagonal elements* of the real-time matrix $\{-+\}$-formalism for the two-point functions, while $\Sigma^{(0)}$ is the self-energy part which is linear in $G - G^{(\mathrm{vac})}$, i.e.,

$$\Sigma^{(0)} = \int \mathrm{d}(34)\Gamma^{(4,\mathrm{vac})}_{12;34}[G - G^{(\mathrm{vac})}]_{34} \quad \text{with} \quad \Gamma^{(4,\mathrm{vac})}_{12;34} = -2\mathrm{i}\frac{\delta^2\Gamma[G]}{\delta G_{12}\delta G_{34}}\bigg|_{T=0}. \tag{4}$$

The abbreviation $\int \mathrm{d}(1 \ldots k)$ stands for the integration over the space-time variables $1, \ldots, k$ with the time variable running along the real-time Schwinger-Keldysh contour. Power counting together with Weinberg's theorem shows that the asymptotic behavior of the various self-energy parts in (3) is $O(p^2)$, $O(p^0)$, and $O(p^{-2})$ (modulo logarithms which are unimportant concerning the convergence properties of the self-consistent diagrams) respectively. Since $\Sigma^{(0)}$ is linear in $G - G^{(\mathrm{vac})}$ it is clear that it obeys itself a linear integral equation in terms of $G^{(r)}$ which behaves asymptotically as $O(p^{-6})$:

$$\Sigma_{12}^{(0)} = \int \mathrm{d}(34) \Lambda_{12;34} G_{34}^{(r)}, \tag{5}$$

where $\Lambda$ has to be determined by the *vacuum Bethe-Salpeter (BS) ladder-equation*

$$\Lambda_{12;34} = \Gamma_{12;34}^{(4,\mathrm{vac})} + \mathrm{i} \int \mathrm{d}(3'4'56) \Gamma_{12;3'4'}^{(4,\mathrm{vac})} G_{3'5}^{(\mathrm{vac})} G_{4'6}^{(\mathrm{vac})} \Lambda_{56;34}. \tag{6}$$

The corresponding momentum integral is logarithmically divergent and a detailed BPHZ-analysis yields a set of renormalized equations in momentum space

$$\begin{aligned}
\Lambda^{(\mathrm{ren})}(0,q) =& \Lambda^{(\mathrm{ren})}(0,0) + \Gamma^{(4,\mathrm{vac})}(0,q) - \Gamma^{(4,\mathrm{vac})}(0,0) \qquad (7)\\
&+ \mathrm{i} \int \frac{\mathrm{d}^4 l}{(2\pi)^4} \Lambda^{(\mathrm{ren})}(0,l) [G^{(\mathrm{vac})}(l)]^2 [\Gamma^{(4,\mathrm{vac})}(l,q) - \Gamma^{(4,\mathrm{vac})}(l,0)],\\
\Lambda^{(\mathrm{ren})}(p,q) =& \Lambda^{(\mathrm{ren})}(0,q) + \Gamma^{(4,\mathrm{vac})}(p,q) - \Gamma^{(4,\mathrm{vac})}(0,q) \qquad (8)\\
&+ \mathrm{i} \int \frac{\mathrm{d}^4 l}{(2\pi)^4} [\Gamma^{(4,\mathrm{vac})}(p,l) - \Gamma^{(4,\mathrm{vac})}(0,l)] [G^{(\mathrm{vac})}(l)]^2 \Lambda^{(\mathrm{ren})}(l,q),
\end{aligned}$$

to be solved in kind of sweep up (7) - sweep down (8) method. Here $\Lambda^{\mathrm{ren}}(0,0)$ defines the renormalization condition. The 2PI property of $\Gamma^{(4,\mathrm{vac})}$ and Weinberg's power counting theorem ensure that the differences of $\Gamma^{(4,\mathrm{vac})}$ appearing in the latter equations are of negative degree of divergence with respect to their integration variable and thus all these integrals are finite. In this pure vacuum equation all quantities are to be read as the time ordered (i.e., the $\{--\}$-components). Since $\Lambda^{(4,\mathrm{vac})}(p,q)$ is of $O(q^0)$ for fixed $p$ the equation (5) is finite since $G^{(r)}(p) = O(p^{-6})$. The finite temperature self-energy is thus rendered finite with *pure vacuum counter terms* by substitution of the renormalized four-point BS-function in (5). The remainder $\Sigma^{(r)}$ is finite by itself after subtracting the explicit vacuum subdivergences, since it contains at least two factors $G - G^{(\mathrm{vac})}$.

Furthermore we could show that also the generating functional can be renormalized by the same counter-terms and finally the renormalized self-energy is defined by Eq. (1), where $\Phi$ is to be read as the *renormalized functional*. An analytic continuation from the real-time to imaginary-time quantities then provides renormalized expressions for thermal equilibrium such as the thermodynamic potentials (like pressure or entropy) which are thermodynamically and dynamically consistent. For details of the complete renormalization formalism see [9].

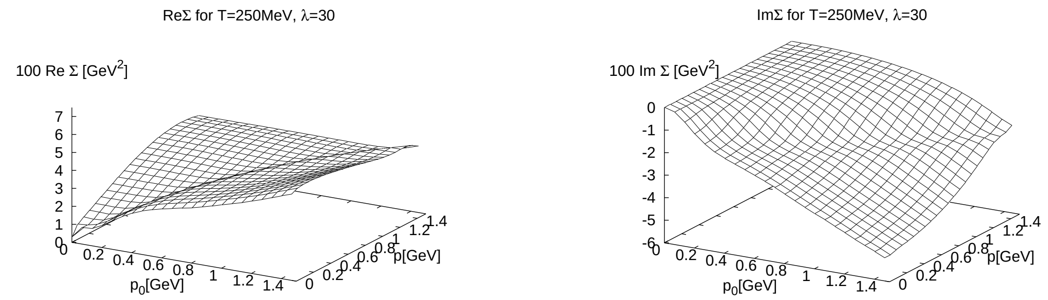



Figure 1: The real and imaginary part of the self-consistent self-energy for $\lambda/24\phi^4$ were calculated. A full calculation without further approximations could be achieved. The main effect of self-consistency is that the higher masses due to the tadpole contribution, which is dominant for the real part of the self-energy lowers the phase-space available for decays into three particles while the growing finite width itself induces a further broadening.

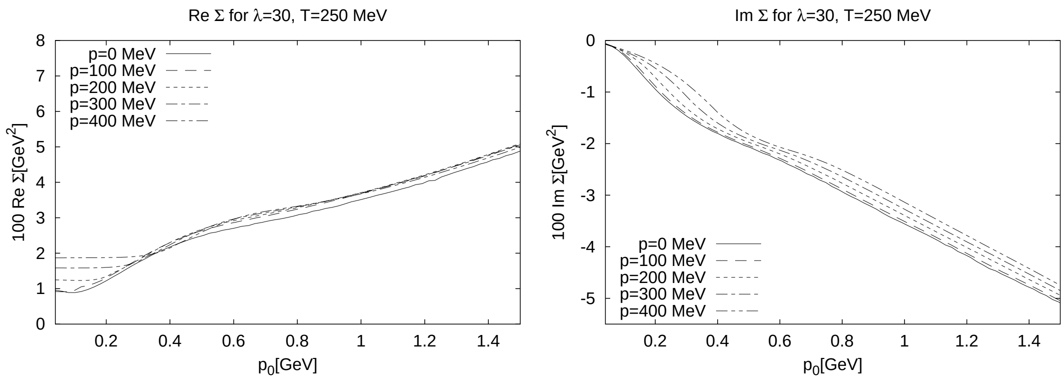



Figure 2: Real (left) and imaginary part (right) of the self-consistent self-energy for $\lambda = 30$, $m = 140\text{MeV}$ and $T = 250\,\text{MeV}$ as a function of $p_0$ for various 3-momenta.

First applications in next to leading order approximation to $\phi^4$-theory, including both, the self-consistent tadpole and the sunset-diagram for the self-energy, were presented [11]. We could demonstrate that not only the UV-problem but also the resulting singularities at the on-shell pole of the vacuum propagator could be tackled numerically. The self-consistent solutions are then obtained iteratively. The finite temperature results for the self-consistent case are shown in Fig. 1 in a 3-dimensional plot over the $(p_0, |\vec{p}|)$-plane illustrating that the entire calculations are performed with the full dependence on energy and momentum on a $200 \times 200$ lattice. Details can be extracted from the cuts shown for a set of selected momenta in Fig. 2.

The main qualitative results are similar for both the perturbative and the self-consistent calculation: In the vacuum and self-consistent pure tadpole case the self-energy shows a threshold cut resulting from the decay into three particles, i.e., $p_0^2 - \vec{p}^2 \geq 9M^2$. Adding the sunset self-energy leads to a spectral width which dissolves this threshold such that the self-energy shows spectral strength (imaginary parts) at all energies. While the growing high-energy tail is related to the decay of virtual bosons into three particles, at finite temperature, as a new component, a

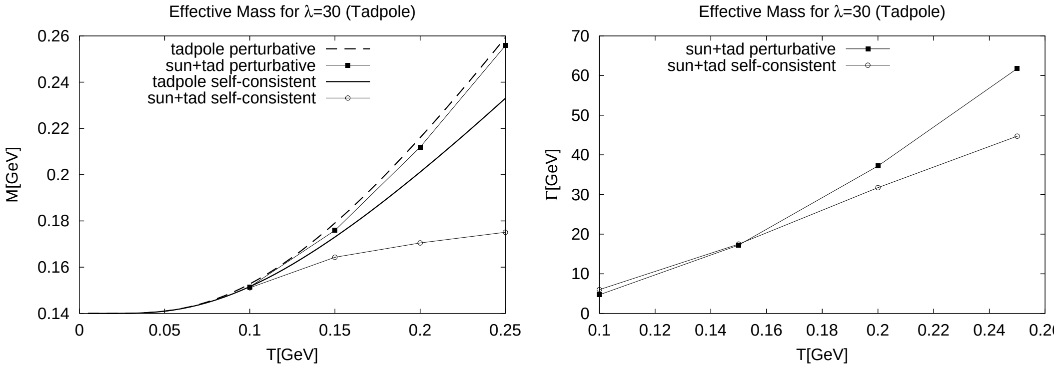



Figure 3: The in-medium effective masses $M$ (left) and spectral widths $\Gamma$ (right) of the particles for the various approximations described in the text as a function of the system's temperature $T$.

low-energy plateau in Im $\Sigma^{\rm R}$ emerges from in-medium scattering processes.

Various balancing effects are encountered for the self-consistent case: For sufficiently large couplings and/or temperatures the self-consistent treatment shows quantitative effects on the width. The finite spectral width itself leads to a further broadening of the width and a smoothing of the structures as a function of energy. This is however counter balanced by the behavior of the real part of the self-energy, which, as discussed below, essentially shifts the in-medium mass upwards. This reduces the available phase space for real processes. With increasing coupling strength $\lambda$ a nearly linear behavior of Im $\Sigma^{\rm R}$ with $p_0$ results implying a nearly constant damping width given by $-$Im $\Sigma^{\rm R}/p_0$.

The overall normalization of the real part of $\Sigma^{\rm R}$ is determined by the renormalization procedure. In this case there are three counter balancing effects. First the tadpole loop shifts the mass to higher values. As the tadpole is less effective for higher masses this effect weakens itself in the self-consistent tadpole treatment, c.f. Figs. 3. However, since the sunset part adds spectral width, it indirectly contributes to the tadpole loop. Since spectral strength at the lower mass side carries higher statistical weights, the tadpole loop in turn leads to a further increase of the mass shift, c.f. the perturbative calculations of sunset & tadpole in Fig. 3.

The direct contributions of the sunset terms to the real part of the self-energy become relevant at higher couplings and temperatures. Then the self-consistency leads to significant effects which contributes to a net down-shift of the real part of the self energy or in-medium mass $M$. The latter effect finally overrules the tadpole shift and indeed leads to an overall negative mass shift compared to the (tadpole dominated) perturbative result. These effects are illustrated in Fig. 3 where the in-medium effective mass $M$ and width $\Gamma$ of the corresponding "quasi-particles" are plotted against the temperature. Thereby $M$ and $\Gamma$ are defined as the quasi-particle energy $M = p_0$ at vanishing real part of the dispersion relation $[p_0^2 - m^2 - {\rm Re}\Sigma(p_0, \vec{p})]_{p=(M,\vec{0})} = 0$ for $\vec{p} = 0$, and through $\Gamma = -{\rm Im}\ \Sigma(p)/p_0|_{p=(M,\vec{0})}$, respectively.

We have shown that it is possible to use the renormalization scheme, proposed in [9], for numerical investigations of the self-consistent approximations for the self-energy derived from the truncated effective action formalism on the 2PI level. Thereby it is very important to isolate the divergent vacuum parts consistently, in particular the implicit or hidden ones, from all convergent and in particular temperature or more generally matter-dependent parts. This could be provided by the ansatz within the real-time formalism of quantum field theory which allows to separate vacuum expressions from genuine finite temperature parts of the propagator and the self-energy. The results promise that the method, which is conserving [5, 12] and thermodynamically consistent, can also be applied for the genuine non-equilibrium case, i.e., in quantum transport [13, 14] or for the solution of the renormalized Kadanoff-Baym equations.

The investigation of the symmetry properties of Φ-derivable approximations is the subject of a forthcoming publication [10]. It is known that in general the symmetries of the classical action which lead to Ward-Takahashi identities for the proper vertex-functions are violated for the self-consistent Dyson resummation for the functions beyond the one-point level, i.e., at the correlator level. The reason is that, although the *functional* $\Gamma$ can be expanded with respect to expansion parameters like the coupling or $\hbar$ (loop expansion) or large-$N$ expansions for $\mathrm{O}(N)$ type models, the solution of the self-consistent equations of motion contains partial contributions to any order of the expansion parameter. This resummation is of course incomplete and violates even crossing symmetry for the vertices involved in the renormalization procedure. This causes problems concerning the Nambu-Goldstone modes [6] in the case of spontaneously broken symmetry or concerning local gauge symmetries [15] when the gauge fields are treated beyond the classical field level, i.e., at the propagator level. In [10] we discuss how to cure these defects by supplementary vertex equations which remain renormalizable following the strategy presented here. Such self-consistent treatments are important for future applications to QCD or hadronic matter problems at finite temperature and finite baryon densities. There further complications arise due to the gauge structure of the gluon or vector mesons [15]. Further applications concern the derivation of renormalized conserving quantum transport equations [13] which permit to treat broad resonances consistently [14].

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