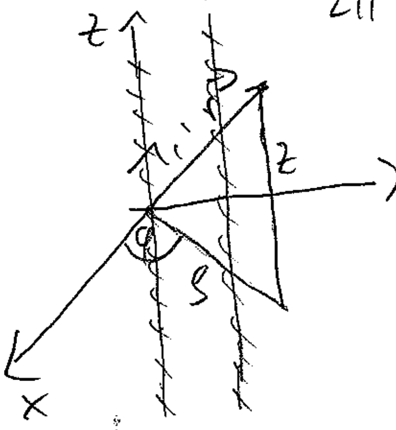
# Problems Chapter 10.

(1) The first wire produces the $\vec{B}$ field

$$\vec{B} = \frac{\mu_0 i}{2\pi\rho}\vec{i}_\varphi$$




$\rho, \varphi, z$ : usual cylinder coordinates

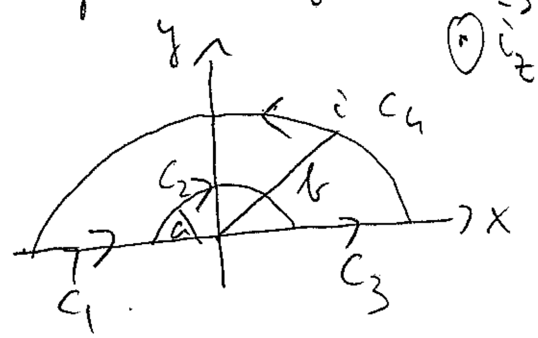
The second wire may be at $\rho = a$ carrying current $i'$. A piece of length $l$ thus feels the force

$$\vec{F} = \frac{\mu_0 i}{2\pi a}\vec{i}_\varphi \times i' l \vec{i}_z$$

$$= -\frac{\mu_0 i i'}{2\pi a}\vec{i}_\rho$$

Thus if the currents go in the same (opposite) direction the force is attracting (repelling)) the wires to each other.

(2)




Parametrizations:

$C_1 : \vec{r}' = x'\vec{i}_x ; x \in (-b, -a) \Rightarrow d\vec{r}' = dx'\vec{i}_x$

$C_2 : \vec{r}' = a\vec{i}_\rho(\varphi) ; \varphi$ from $\pi \to 0 \Rightarrow d\vec{r}' = a d\varphi \vec{i}_\varphi$

$C_3 : \vec{r}' = x'\vec{i}_x ; x \in (a, b) \Rightarrow d\vec{r}' = dx' = \vec{i}_x$

$C_4 : \vec{r}' = b\vec{i}_\rho(\varphi) ; \varphi \in (0, \pi) \Rightarrow d\varphi = b d\varphi \vec{i}_\rho$

$$\vec{B}(\vec{r}=0) = \int_{C_1+C_2+C_3+C_4} d\vec{r}' \times \frac{\mu_0 i}{4\pi} \frac{-\vec{r}'}{|\vec{r}'|^3}$$

The lines $C_1$ and $C_3$ do not contribute since $d\vec{r}' \times \vec{r}' = 0$

$$\Rightarrow \vec{B}_2(\vec{r}=0) = \int_{\pi}^{0} d\varphi\, a\, \vec{i}_\varphi \times \frac{\mu_0 i}{4\pi} \frac{(-\vec{i}_\varrho(\varphi'))}{a^2} = -\frac{\mu_0 i}{4a} \vec{i}_z$$

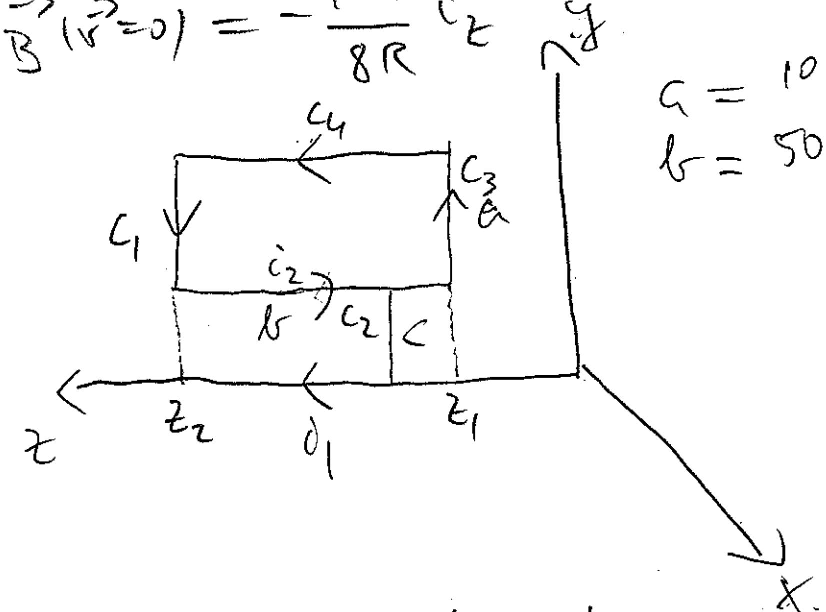
$$\vec{B}_4(\vec{r}=0) = \int_{0}^{\pi} d\varphi\, b\, \vec{i}_\varphi \times \frac{\mu_0 i}{4\pi} \frac{(-\vec{i}_\varrho(\varphi'))}{b^2} = \frac{\mu_0 i}{4b} \vec{i}_z$$

$$\Rightarrow \vec{B}(\vec{r}=0) = -\frac{\mu_0 i}{4}\left(\frac{1}{a} - \frac{1}{b}\right)\vec{i}_z$$

For $a = R$ and $b = 2R$

$$\vec{B}(\vec{r}=0) = -\frac{\mu_0 i}{8R}\vec{i}_z$$

(3)




$a = 10$ cm
$b = 50$ cm

In the $yz$ plane we have for $y > 0$

$$\vec{B} = -\frac{i_1 \mu_0}{2\pi \varrho_y y}\vec{i}_x \quad \left(\text{since } \vec{i}_\varphi(\varphi = \tfrac{\pi}{2}) = -\vec{i}_x\right)$$

Forces on $C_1, \ldots, C_4$:

$C_1: \vec{r} = y\vec{i}_y + z_2\vec{i}_z$ with $y$ running from $a+c$ to $c$

$$\vec{F}_1 = \int_{a+c}^{c} dy\, \vec{i}_y\, i_2 \times \left(-\frac{i_1\mu_0}{2\pi y}\vec{i}_x\right)$$

$\vec{F}_1 = \vec{i}_z \frac{i_1 i_2 \mu_0}{2\pi} \int_{a+c}^{c} dy \frac{1}{y} = \frac{i_1 i_2}{2\pi\mu_0} \ln\left(\frac{c}{a+c}\right) \vec{i}_z$ ③

$= -\frac{i_1 i_2 \mu_0}{2\pi} \ln\left(\frac{a+c}{c}\right) \vec{i}_z$

Since for $C_3$ we have the same integral in the opposite direction, because $\vec{B}$ does not depend on $z$, $\vec{F}_3 = -\vec{F}_1$.

$C_2: \vec{F}_2 = -b i_2 \vec{i}_z \times \left(-\frac{\mu_0 i_1}{2\pi c} \vec{i}_x\right)$

$= \frac{\mu_0 i_1 i_2 b}{2\pi c} \vec{i}_y$

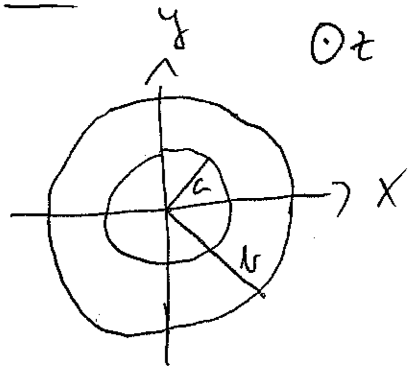
$C_4: \vec{F}_4 = -\frac{\mu_0 i_1 i_2 b}{2\pi (a+c)} \vec{i}_y$

$\Rightarrow \vec{F}_{total} = \frac{\mu_0 i_1 i_2 b}{2\pi} \left(\frac{1}{c} - \frac{1}{a+c}\right) \vec{i}_y = 10^{-5} N \vec{i}_y$

(4) See lectures

(5) — " —

(6)




$\vec{j} = \begin{cases} \frac{i}{\pi(b^2-a^2)} \vec{i}_z & \text{for } a<\varrho<b \\ 0 & \text{elsewhere} \end{cases} \Rightarrow \vec{B} = B_\varphi \vec{i}_\varphi$

Ampère's law with circles of radius $\varrho$:

$\oint_C d\vec{r}\, \vec{B} = 2\pi \varrho B_\varphi = \mu_0 i_{\text{inside}}$

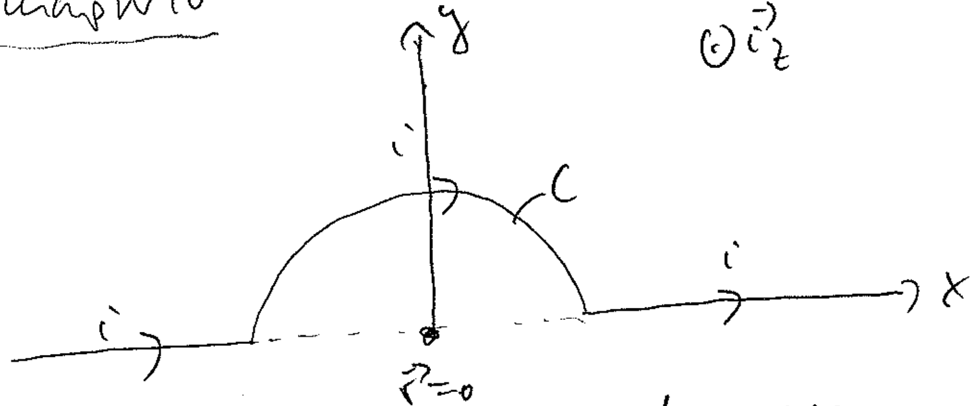
$$\Rightarrow B_\varphi = 0 \text{ for } 0 < \rho < a$$

$$B_\varphi = \frac{\mu_0 i}{2\pi\rho} \frac{\pi(\rho^2 - a^2)}{\pi(b^2 - a^2)} = \frac{\mu_0 i}{2\pi\rho} \frac{\rho^2 - a^2}{b^2 - a^2} \text{ for } a < \rho < b$$

$$B_\varphi = \frac{\mu_0 i}{2\pi\rho} \text{ for } \rho > b$$

Exercises Chapter 10 

(1)



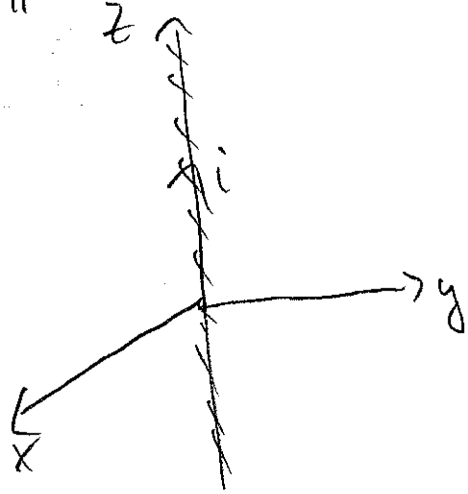

Biot-Savart law ⇒ no contribution from wires along x axis.

For semi circle:

$C: \vec r\,' = R\,\vec i_\varrho(\varphi)$ with $\varphi$ from $\pi \to 0$; $d\vec r\,' = R\,\vec i_\varphi(\varphi)$

$$\vec B = \frac{\mu_0 i}{4\pi}\int_\pi^0 d\varphi\, R\, \vec i_\varphi(\varphi) \times \frac{-\vec i_\varrho(\varphi)}{R^2} = -\frac{\mu_0 i}{4R}\vec i_z$$

(2)




$$\vec B = \frac{\mu_0 i}{2\pi\varrho}\vec i_\varphi$$

$$\vec F = q\,\vec v_0 \times \vec B$$

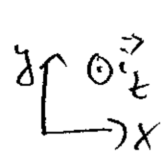
(a) $\vec v_0 = v_0 \vec i_z \Rightarrow \vec F = \frac{q\mu_0 i v_0}{2\pi d}(-\vec i_\varrho)$

(b) $\vec v_0 = v_0 \vec i_\varrho \Rightarrow \vec F = \frac{q\mu_0 i v_0}{2\pi d}\vec i_z$

(c) $\vec v_0 = v_0 \vec i_\varphi \Rightarrow \vec F = 0$

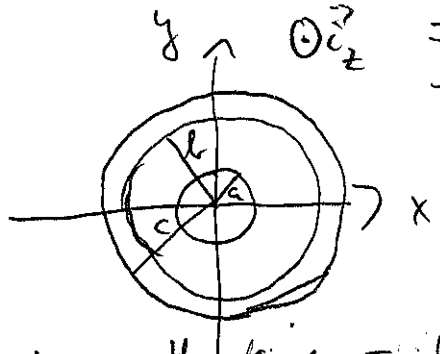
(3) let $\vec{i}_z$ come out of the plane. Then at the centre wire:

$$\vec{B} = \vec{B}_1 + \vec{B}_2 = \frac{\mu_0}{2\pi}\left(-\frac{i_1}{l_1} + \frac{i_3}{l_2}\right)\vec{i}_z$$



$$\vec{F} = i_2 l\, \vec{i}_x \times \vec{B} = \frac{\mu_0 i_2 l}{2\pi}\left(\frac{i_1}{l_1} - \frac{i_3}{l_2}\right)\vec{i}_y$$

(4) $\vec{B} = B_\varphi \vec{i}_\varphi$



Ampère's law with ~~area~~ = circular disk of radius $\rho$

$$\vec{j} = \begin{cases} -\dfrac{i}{\pi a^2}\vec{i}_z & \text{for } 0 < \rho < a \\ +\dfrac{i}{\pi(c^2-b^2)}\vec{i}_z & \text{for } b < \rho < c \\ 0 & \text{elsewhere} \end{cases}$$

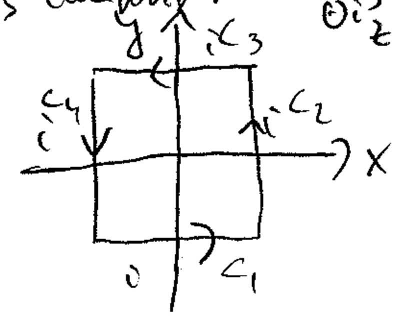
$$\oint_{\partial D_\rho} d\vec{r}\cdot\vec{B} = \int_{D_\rho} d\vec{S}\cdot\vec{j}$$

$$2\pi\rho B_\varphi = \mu_0 i_{\text{inside}} = \mu_0 \begin{cases} -\dfrac{i\pi\rho^2}{\pi a^2} & \text{for } 0 < \rho < a \\ -i & \text{for } a < \rho < b \\ -i + \dfrac{i\pi(\rho^2 - b^2)}{\pi(c^2-b^2)} & \text{for } b < \rho < c \\ 0 & \text{for } \rho > c \end{cases}$$

$$B_\varphi = \frac{\mu_0 i}{2\pi\rho} \cdot \begin{cases} -\rho^2/a^2 & \text{for } 0<\rho<a \\ -1 & \text{for } a<\rho<b \\ -\frac{c^2-\rho^2}{c^2-b^2} & \text{for } b<\rho<c \\ 0 & \text{for } \rho>c \end{cases}$$

B is always tangent to the circles in clockwise direction!

(5)




Biot-Savart Law

$c_1: \vec{r}' = -\frac{l}{2}\vec{i}_y + x\vec{i}_x \; ; x \in (-\frac{l}{2}, \frac{l}{2})$

$$\vec{B}_1 = \frac{\mu_0 i}{4\pi}\int_{-l/2}^{l/2} dx\, \vec{i}_x \times \frac{l}{2}\vec{i}_y \frac{1}{\left(x^2+\frac{l^2}{4}\right)^{3/2}}$$

$$= \frac{\mu_0 i}{4\pi}\frac{l}{2}\int_{-l/2}^{l/2} dx \frac{\vec{i}_z}{\left(x^2+\frac{l^2}{4}\right)^{3/2}} = \frac{\mu_0 i l}{4\pi}\int_0^{l/2} dx \frac{\vec{i}_z}{\left(x^2+\frac{l^2}{4}\right)^{3/2}}$$

$$\int dx \frac{1}{(x^2+a^2)^{3/2}} = \int du\, a\cosh u \frac{1}{a^3\cosh^3 u} = \frac{1}{a^2}\int du \frac{1}{\cosh^2 u} = \frac{\tanh u}{a^2}$$

$x = a\sinh u \Rightarrow dx = a\cosh u\, du$

$a = \frac{l}{2} \Rightarrow x = a \Rightarrow \sinh u = 1 \Rightarrow \tanh u = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

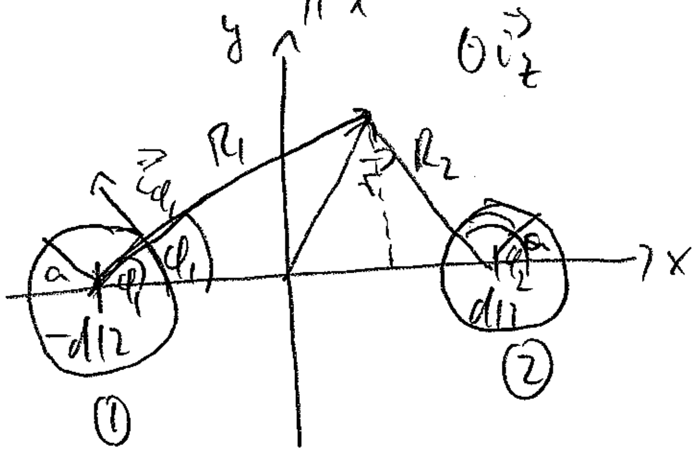
$\Rightarrow x = 0 \Rightarrow \sinh u = 0 \Rightarrow \tanh u = 0$

$$\Rightarrow \vec{B}_1 = \frac{\mu_0 i l}{4\pi}\,\frac{\sqrt{2}}{2}\cdot\frac{4}{l^2}\,\vec{i}_z = \frac{\sqrt{2}\,\mu_0 i}{2\pi l}\,\vec{i}_z$$

Everything else is totally symmetric and gives thus the same contribution to $\vec{B}$

$$\Rightarrow \vec{B} = 4\vec{B}_1 = \frac{2\sqrt{2}\,\mu_0 i}{\pi l}\,\vec{i}_z$$

(b)




(a) $\vec{r}$ outside of wires

$$\vec{B}_1 = \frac{i\mu_0 \vec{i}_{\varphi_1}}{2\pi R_1} = \frac{i\mu_0}{2\pi R_1}\left(-\sin\varphi_1\,\vec{i}_x + \cos\varphi_1\,\vec{i}_y\right)$$

$$= \frac{i\mu_0}{2\pi R_1}\left(-\frac{y}{R_1}\,\vec{i}_x + \frac{x+\frac{d}{2}}{R_1}\,\vec{i}_y\right)$$

$$R_1 = \left[\left(x+\frac{d}{2}\right)^2 + y^2\right]^{1/2}$$

$$\vec{B}_2 = \frac{i\mu_0}{2\pi R_2}\left(-\frac{y}{R_2}\,\vec{i}_x + \frac{x-\frac{d}{2}}{R_2}\,\vec{i}_y\right)$$

$$R_2 = \left[\left(x-\frac{d}{2}\right)^2 + y^2\right]^{1/2}$$

$$\vec{B} = \vec{B}_1 + \vec{B}_2$$

Look for $x, y$ such that $\vec{B} = 0$

x direction

$$-\frac{y}{R_1^2} - \frac{y}{R_2^2} = 0 \quad \text{since } R_1, R_2 > 0 \Rightarrow y = 0$$

$\Rightarrow$ y-direction

$$\frac{x + \frac{d}{2}}{R_1^2} + \frac{x - \frac{d}{2}}{R_2^2} = 0$$

$$R_1 = \left|x + \frac{d}{2}\right| \; ; \; R_2 = \left|x - \frac{d}{2}\right|$$

Cases:

(a) $x + \frac{d}{2} > 0,\ x - \frac{d}{2} > 0 \Rightarrow$ no solution

(b) $x + \frac{d}{2} > 0,\ x - \frac{d}{2} < 0 \Rightarrow \frac{1}{x + \frac{d}{2}} + \frac{1}{x - \frac{d}{2}} = 0$

$$\Rightarrow x + \frac{d}{2} = -\left(x - \frac{d}{2}\right) \Rightarrow x = 0$$

Since for $x = 0$ the condition is fulfilled that's a solution (and it's outside the wires too!) ☺

(c) $x + \frac{d}{2} < 0 \; ; \; x - \frac{d}{2} < 0 \Rightarrow$ no solution either

Only solution outside wires: $\boxed{x = y = 0}$

Point inside right wire:

$\vec{B}_1$ : as before

$$\vec{B}_2 = \frac{i\mu_0}{2\pi}\frac{R_2}{a^2}\vec{i}_{\varphi_2}$$

$$= \frac{i\mu_0}{2\pi}\frac{\cancel{R_2}}{a^2}\left(-\frac{y}{\cancel{R_2}}\vec{i}_x + \frac{x-\frac{d}{2}}{\cancel{R_2}}\vec{i}_y\right)$$

$$\vec{B}_1 + \vec{B}_2 = 0$$

x-direction

$$-\frac{y}{R_1^2} - \frac{y}{a^2} = 0 \Rightarrow y = 0$$

y-direction

$$\frac{x+\frac{d}{2}}{R_1^2} + \frac{x-\frac{d}{2}}{a^2} = 0$$

Since $x + \frac{d}{2} > 0$ ; $R_1 = x + \frac{d}{2}$

$$\frac{1}{x+d/2} + \frac{x-d/2}{a^2} = 0$$

$$a^2 + x^2 - \frac{d^2}{4} = 0 \Rightarrow x = \sqrt{\frac{d^2}{4} - a^2} = \frac{1}{2}\sqrt{d^2-4a^2}$$

which is inside the right wire since $a < \frac{d}{2}$ ☺

Comparison to book: Distance from center of right wire: $x' = \frac{d}{2} - x$ ✓

Solution for point in left wire by symmetry: $x = -\frac{1}{2}\sqrt{d^2-4a^2}$, $y = 0$.