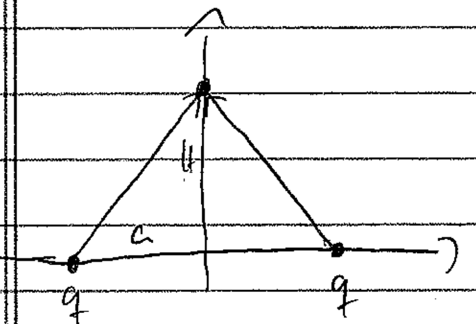
Problems Chapter 2

(1)



$$\vec{E} = \frac{q}{4\pi\varepsilon_0}\frac{1}{(a^2+H^2)^{3/2}}\cdot$$

$$\cdot\left[a\,\vec{i}_x + H\,\vec{i}_y + (-a)\,\vec{i}_x + H\,\vec{i}_y\right]$$

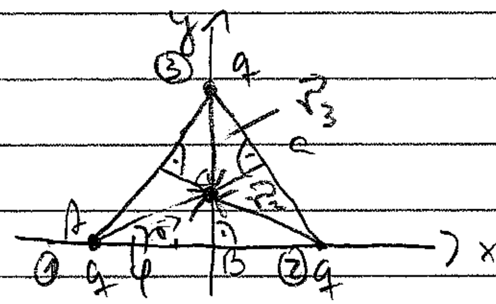
$$\vec{E} = \frac{q}{4\pi\varepsilon_0}\frac{1}{(a^2+H^2)^{3/2}}\,2H\,\vec{i}_y$$

$$\vec{E} = \frac{qH}{2\pi\varepsilon_0(a^2+H^2)^{3/2}}\,\vec{i}_y$$

This holds true for any $y = H$ (for a point on the y-axis!)

$$\vec{E}(y\,\vec{i}_y) = \frac{q\,y}{2\pi\varepsilon_0(a^2+y^2)^{3/2}}\,\vec{i}_y$$

(2)



$\vec{E} = \vec{E}_1 + \vec{E}_2 + \vec{E}_3$ where $\vec{E}_k$ is the electric field caused by charge number k ($k \in \{1,2,3\}$)

$$\vec{E}_1 = \frac{q}{4\pi\varepsilon_0}\frac{\vec{r}_1}{|\vec{r}_1|^3}\ ;\ \vec{r}_1 = x_1\,\vec{i}_x + y_1\,\vec{i}_y$$

Obviously: $x_1 = \frac{a}{2}$

Further we use triangle ABC ($\varphi = 30^\circ$ from symmetry)

$$\frac{y_1}{x_1} = \tan\varphi = \frac{\sqrt{3}}{3} \Rightarrow y_1 = \frac{\sqrt{3}}{3}\,\frac{a}{2} = \frac{\sqrt{3}}{6}\,a$$

From symmetry, it's easy to read off the other components:

$$|\vec{r}_1| = r_1 = \sqrt{\frac{a^2}{4} + \frac{3}{36}a^2} = \sqrt{\frac{1}{12} + \frac{1}{4}}\, a = \frac{1}{\sqrt{3}}a = \frac{\sqrt{3}}{3}a$$

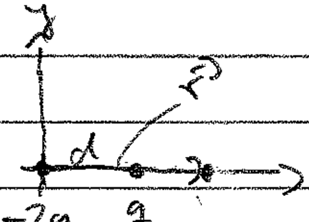
$$\vec{E}_2 = \frac{q}{4\pi\varepsilon_0}\frac{\vec{r}_2}{r_2^3} = \frac{q}{4\pi\varepsilon_0 r_2^3}\left(-\frac{a}{2}\vec{i}_x + \frac{\sqrt{3}}{6}a\,\vec{i}_y\right)$$

$$\vec{E}_3 = \frac{q}{4\pi\varepsilon_0}\frac{\vec{r}_3}{r_3^3} = \frac{q}{4\pi\varepsilon_0}\frac{1}{r_3^3}\left(-\frac{\sqrt{3}}{3}a\,\vec{i}_y\right)$$

Further we have from symmetry $r_1 = r_2 = r_3 = \frac{\sqrt{3}}{3}a$

$\Rightarrow \vec{E} = \sum_{k=1}^{3}\vec{E}_k = 0$ (which we would have guessed from the very beginning!)

(3)



$\vec{r} = x\vec{i}_x$

$$\vec{E} = \frac{-2q}{4\pi\varepsilon_0}\frac{1}{x^2}\vec{i}_x + \frac{q}{4\pi\varepsilon_0}\frac{1}{(x-d)^2}\,\mathrm{sign}(x-d)\,\vec{i}_x$$

$$= \frac{q}{4\pi\varepsilon_0}\left[\frac{\mathrm{sign}(x-d)}{(x-d)^2} - \frac{2}{x^2}\right]\vec{i}_x \overset{!}{=} 0$$

$$\Rightarrow \mathrm{sign}(x-d) = \frac{2(x-d)^2}{x^2} \geq 0$$

$$\Rightarrow x \geq d$$

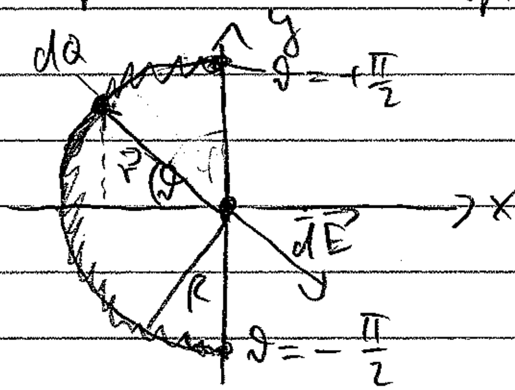
$$\Rightarrow x^2 = 2(x-d)^2 \Rightarrow x = \sqrt{2}(x-d)$$

$$\Rightarrow x = \frac{d}{\sqrt{2}-1} = \frac{d(\sqrt{2}+1)}{(\sqrt{2}-1)(\sqrt{2}+1)} = \frac{d(\sqrt{2}+1)}{2-1} = d(\sqrt{2}+1)$$

(4) Just take solution to problem 6 and divide by q. So the answer is

$$\vec{E} = \frac{Q}{4\pi\varepsilon_0\, a(a+L)} \left(-\vec{i}_x\right)$$

(5) Here the same idea as in the previous problem applies. You get the electric field by "summing" over all infinitesimal charges (see figure)



$$dQ = \frac{Q}{\pi}\, d\vartheta$$

$$\vec{E}(0) = \int_{\vartheta=-\frac{\pi}{2}}^{\vartheta=+\frac{\pi}{2}} \frac{dQ}{4\pi\varepsilon_0} \frac{1}{R^2} \left(\frac{-\vec{r}}{R}\right)$$

$$= \frac{Q}{4\pi^2 \varepsilon_0 R^2} \int_{-\pi/2}^{\pi/2} d\vartheta \left(\cos\vartheta\, \vec{i}_x + \sin\vartheta\, \vec{i}_y\right)$$

Note that $\cos\vartheta \geq 0$ for $-\frac{\pi}{2} \leq \vartheta \leq \frac{\pi}{2}$ and $\sin\vartheta = \operatorname{sign}\vartheta$ in $-\frac{\pi}{2} \leq \vartheta \leq \frac{\pi}{2}$ $\Rightarrow$ The vector components have the correct sign everywhere on the half circle.

Now:

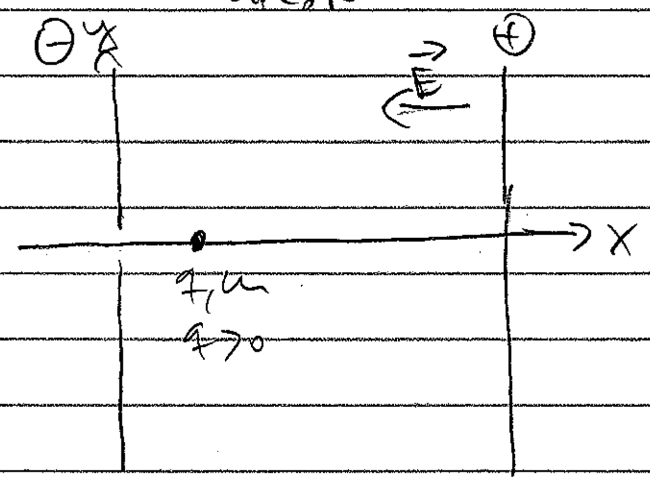
$$\int_{-\pi/2}^{\pi/2} d\vartheta \cos\vartheta = \sin\vartheta \Big|_{-\pi/2}^{\pi/2} = 2\sin\left(\frac{\pi}{2}\right) = 2$$

$$\int_{-\pi/2}^{\pi/2} d\vartheta \sin\vartheta = 0 \quad \text{(since sin is an odd function)}$$

Thus

$$\vec{E}(0) = \frac{Q}{2\pi\varepsilon_0 R^2}\,\vec{i}_x$$

(b)



$\vec{E} = -E\,\vec{i}_x$

$\vec{E}$ must point in negative x-direction, because we want the charge be decelerated.
Newton's 2nd law for the x component is enough to solve the problem

$$m\ddot{x} = -qE$$

As shown in the lecture on Friday (Jan 27):

$$x = -\frac{qE}{2m}t^2 + v_0 t$$

$$\Rightarrow \dot{x} = v = -\frac{qE}{m}t + v_0$$

The particle stops when $v=0$:

$$v = -\frac{qE}{m} t_0 + v_0 \stackrel{!}{=} 0$$

$$\Rightarrow t_0 = \frac{v_0 \cdot m}{qE}$$

Now this should be when $x=L$

$$\Rightarrow -\frac{qE}{2m}\left(\frac{v_0 m}{qE}\right)^2 + \frac{v_0^2 m}{qE} \stackrel{!}{=} L$$

$$\Rightarrow \frac{v_0^2 m}{2qE} = L$$

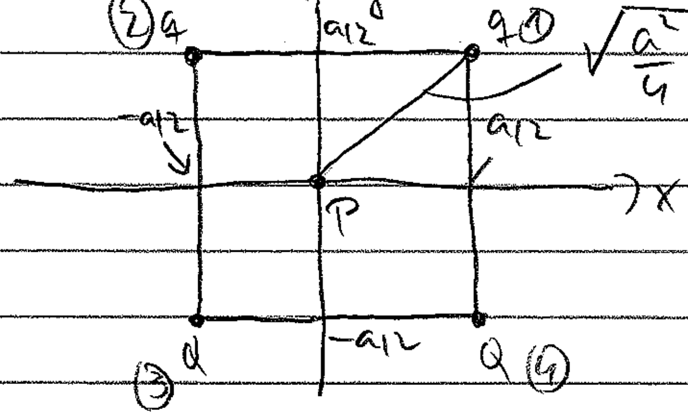
$$\Rightarrow \boxed{E = \frac{v_0^2 m}{2qL}}$$

Exercises Chapter 2

(1) You just need to use the formula from the lecture (and put on the signs of the charges as given!)

$$\vec{E} = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{(L+3a)^2} + \frac{1}{(L+2a)^2} - \frac{1}{(L+a)^2} - \frac{1}{L^2}\right)\vec{i}_x$$

(2) First choose convenient coordinates. My suggestions:



$$\sqrt{\frac{a^2}{4}\cdot 2} = \frac{a}{\sqrt{2}} = \frac{\sqrt{2}\,a}{2}$$

Then we want $\vec{E}$ on the origin

$$\vec{E} = \frac{1}{4\pi\varepsilon_0\left(\frac{\sqrt{2}a}{2}\right)^3}\Big[\underbrace{-q\left(\frac{a}{2}\vec{i}_x + \frac{a}{2}\vec{i}_y\right)}_{①}$$

$$\underbrace{-q\left(-\frac{a}{2}\vec{i}_x + \frac{a}{2}\vec{i}_y\right)}_{②}$$

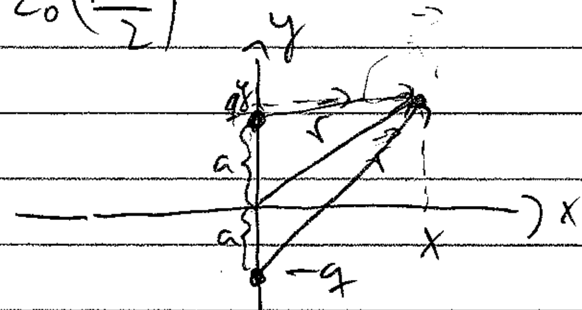
$$-Q\left(-\frac{a}{2}\vec{i}_x - \frac{a}{2}\vec{i}_y\right)$$

$$-Q\left(\frac{a}{2}\vec{i}_x - \frac{a}{2}\vec{i}_y\right)\Big]$$

$$\vec{E} = \frac{1}{4\pi\varepsilon_0\left(\frac{\sqrt{2}a}{2}\right)^3}\left[-aq\,\vec{i}_y + aQ\,\vec{i}_y\right]$$

$$= \frac{a}{4\pi\varepsilon_0\left(\frac{\sqrt{2}a}{2}\right)^3}(Q-q)\,\vec{i}_y = -28,64 \cdot 10^4 \frac{N}{C}\,\vec{i}_y$$

(3)



$$\vec{r} = x\vec{i}_x + y\vec{i}_y$$

Just use our standard formula.

$$\vec{E}(\vec{r}) = \frac{q}{4\pi\varepsilon_0}\left[\frac{\vec{r} - a\vec{i}_y}{[x^2+(y-a)^2]^{3/2}} - \frac{\vec{r} + a\vec{i}_y}{[x^2+(y+a)^2]^{3/2}}\right]$$

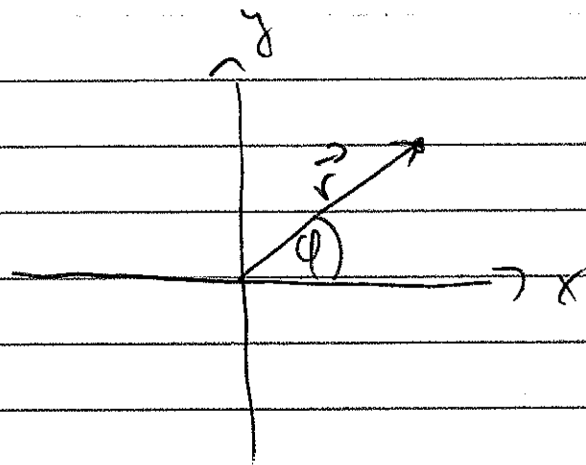
The components are now simply read off from $\vec{r} = x\vec{i}_x + y\vec{i}_y$:

$$E_x = \frac{q}{4\pi\varepsilon_0}\left[\frac{x}{[x^2+(y-a)^2]^{3/2}} - \frac{x}{[x^2+(y+a)^2]^{3/2}}\right]$$

$$E_y = \frac{q}{4\pi\varepsilon_0}\left[\frac{y-a}{[x^2+(y-a)^2]^{3/2}} - \frac{y+a}{[x^2+(y+a)^2]^{3/2}}\right]$$

(4) Same idea as exercise 5: Integrate over charge elements. This time we use polar coordinates in the usual convention:

$$\vec{r} = x\vec{i}_x + y\vec{i}_y = r\cos\varphi\,\vec{i}_x + r\sin\varphi\,\vec{i}_y$$



$r = |\vec{r}|$

$0 \le \varphi < 2\pi$

$$\vec{E}(0) = \int_{\varphi=0}^{\varphi=2\pi} \frac{dQ}{4\pi\varepsilon_0}\left(\frac{-\vec{r}}{r^3}\right)$$

$$= \left[\int_0^{\pi} d\varphi \frac{Q}{4\pi^2\varepsilon_0 R^2}\left(-\vec{i}_x \cos\varphi - \vec{i}_y \sin\varphi\right)\right.$$

$$\left. - \int_{\pi}^{2\pi} d\varphi \frac{Q}{4\pi^2\varepsilon_0 R^2}\left(-\vec{i}_x \cos\varphi - \vec{i}_y \sin\varphi\right)\right]$$

Now:

$$\int_0^{\pi} d\varphi \cos\varphi = \sin\varphi\Big|_0^{\pi} = 0$$

$$\int_0^{\pi} d\varphi \sin\varphi = -\cos\varphi\Big|_0^{\pi} = 2$$

$$\int_{\pi}^{2\pi} d\varphi \cos\varphi = \sin\varphi\Big|_{\pi}^{2\pi} = 0$$

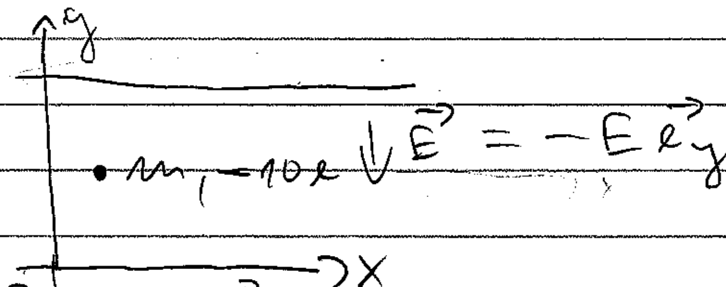
$$\int_{\pi}^{2\pi} d\varphi \sin\varphi = -\cos\varphi\Big|_{\pi}^{2\pi} = -2$$

$$\vec{E}(0) = \frac{Q}{4\pi^2 \varepsilon_0 R^2}\left[-2\vec{i}_y - 2\vec{i}_y\right]$$

$$= -\frac{Q}{\pi^2 \varepsilon_0 R^2}\vec{i}_y$$

For $Q > 0$ that's straight down.

(5)



$$\vec{F}_{el} = q\cdot\vec{E} = -10e\left(-E\vec{e}_y\right) = 10eE\vec{e}_y$$

(resp as it should be: note that the electron is negatively charged with $q_{el} = -e = -1.6\cdot 10^{-19}\,C$. Thus the force in an $\vec{E}$ field points in the opposite direction than the $\vec{E}$ field, because $\vec{F} = q\vec{E}$!)

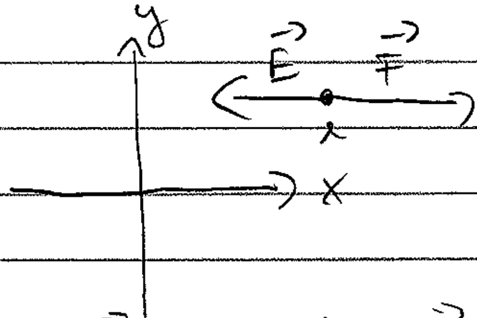
$$\vec{F}_G = -mg\,\vec{i}_y$$

$$\vec{F}_{el} + \vec{F}_G = (10eE - mg)\,\vec{i}_y \overset{!}{=} 0$$

$$\Rightarrow E = \frac{mg}{10e} = 1.225\cdot 10^6\,\frac{N}{C}$$

(using $g = 9.8\,\frac{m}{s^2}$)

6/



$\vec{E} = -E\vec{i}_x$

$$\vec{F} = q\vec{E} = -e(-E\vec{i}_x) = eE\vec{i}_x = F\vec{i}_x$$

(pointing to the right $F = 4.8\cdot10^{-15}\,N$

Our general solution in class gives

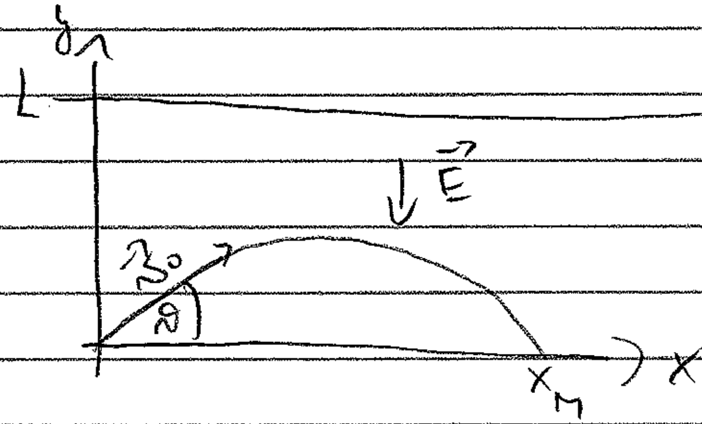
$$x = \frac{eE}{2m} t^2 \vec{i}_x$$

$$\frac{eE}{2m} t_L^2 \overset{!}{=} L \quad (=0.1\,m)$$

$$t_L = \sqrt{\frac{2mL}{eE}}$$

$$v_L = \frac{eE}{m} t_L = \sqrt{\frac{2eEL}{m}} = 3.25\cdot10^7 \frac{m}{s}$$

(7)



$\vec{E} = -E\vec{i}_y$

It must be meant $q>0$

Our general solution in class tells us

$$x = v_0 \cos\vartheta\, t$$

$$y = -\frac{qE}{2m} t^2 + v_0 \sin\vartheta\, t$$

$y = 0$ for $t = 0$ and t_2:

$$-\frac{qE}{2m} t_M + v_0 \sin\vartheta = 0$$

$$\Rightarrow t_M = \frac{2m v_0 \sin\vartheta}{qE}$$

$$\Rightarrow x_M = v_0 \cos\vartheta \, t_M = \frac{2m v_0^2 \sin\vartheta \cos\vartheta}{qE}$$

maximal height is reached when

$$v_y = \dot{y} = -\frac{qEt_i}{m} + v_0 \sin\vartheta = 0$$

$$\Rightarrow t_{max} = \frac{m v_0 \sin\vartheta}{qE} \quad \left(= \frac{t_M}{2}\right)$$

$$\Rightarrow y_{max} = -\frac{qE}{2m} t_{max}^2 + v_0 \sin\vartheta \, t_{max}$$

$$= -\frac{qE}{2m} \frac{m^2 v_0^2 \sin^2\vartheta}{q^2 E^2} + v_0 \sin\vartheta \frac{m v_0 \sin\vartheta}{qE}$$

$$\Rightarrow y_{max} = \frac{m v_0^2 \sin^2\vartheta}{2qE} < L$$