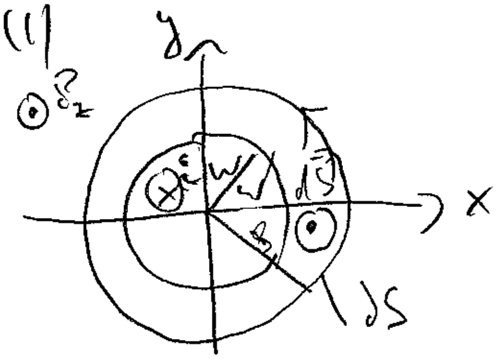
Exam 3 Spring 2008

(1)



$\vec{B} = B(\rho)\,\vec{i}_\varphi$ (in usual cylinder coordinates)

Ampère's law with circle of radius ρ:

$$\oint_{\partial S} d\vec{r}\cdot\vec{B} = \mu_0 \int_S d\vec{S}\,\vec{j}$$

$$d\vec{S} = \rho\, d\rho\, d\varphi\, \vec{i}_z \qquad \Big| \qquad d\vec{r} = \rho\, d\varphi\, \vec{i}_\varphi$$

$$\vec{j} = \begin{cases} -\dfrac{i}{\pi w^2}\,\vec{i}_z & \text{for } \rho < w \\ 0 & \text{for } \rho > w \end{cases}$$

$$\Rightarrow \oint_{\partial S} d\vec{r}\cdot\vec{B} = 2\pi\rho\, B(\rho) = -\frac{\mu_0 i}{\pi w^2}\cdot\begin{cases} \pi\rho^2 & \text{if } \rho < w \\ \pi w^2 & \text{if } \rho > w \end{cases}$$

$$\Rightarrow B(\rho) = -\frac{\mu_0 i}{2\pi w^2 \rho}\begin{cases} \rho^2 & \text{for } \rho < w \\ w^2 & \text{for } \rho > w \end{cases}$$

$$B(\rho = w) = -\frac{\mu_0 i}{2\pi w}$$

Assume $x > 0$. Then $\varphi = 0$ and $x = \rho$

(a) $0 < x < w$

$$\frac{\cancel{\mu_0 i}\, x}{\cancel{2\pi} w^2} = \frac{1}{2}\frac{\cancel{\mu_0 i}}{\cancel{2\pi w}} \Rightarrow x = \frac{w}{2} < w \Rightarrow \text{solution}$$

(b) $x > w$

$$\frac{\cancel{\mu_0 i}}{\cancel{2\pi} x} = \frac{1}{2}\frac{\cancel{\mu_0 i}}{\cancel{2\pi} w} \Rightarrow x = 2w > w \Rightarrow \text{solution}$$

Since the magnitude

$$|\vec{B}| = \frac{\mu_0 i}{2\pi w^2 \rho} \begin{cases} \rho^2 & \text{for } \rho < w \\ w^2 & \rho > w \end{cases}$$

is independent of φ. The other solutions on the x axis are at $\varphi = \pi$ i.e.

$$x = -\frac{w}{2} \text{ and } x = -2w$$

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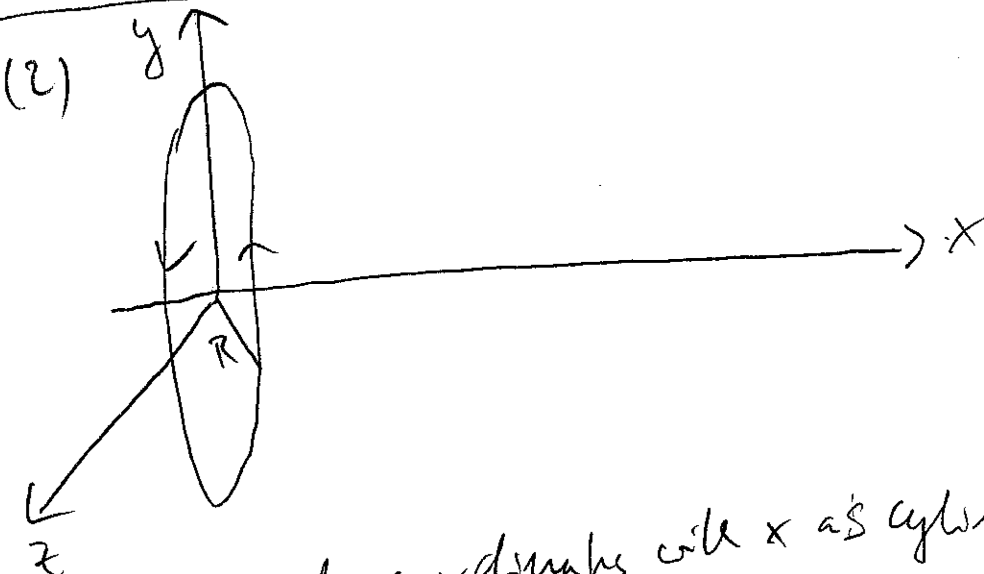
(2)



(a) Use cylinder coordinates with x as cylinder axis

Then we set

$$C: \vec{r}'(\varphi) = R\,\vec{i}_\rho(\varphi) \Rightarrow d\vec{r}' = R\,\vec{i}_\varphi(\varphi)\,d\varphi$$

$$\vec{r} = x\,\vec{i}_x$$

in Biot-Savart's law:

$$\vec{B}(\vec{r}) = \frac{\mu_0 i}{4\pi} \int_C d\vec{r}' \times \frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^2}$$

$$\vec{r}-\vec{r}' = x\,\vec{i}_x - R\,\vec{i}_\rho \Rightarrow d\vec{r}' \times (\vec{r}-\vec{r}') = d\vec{r}' \times (x\,\vec{i}_x - R\,\vec{i}_\rho)$$

$$= R\,d\varphi\,[x\,\vec{i}_\rho(\varphi) + R\,\vec{i}_x]$$

$$|\vec{r}-\vec{r}'| = \sqrt{x^2+R^2}$$

Since $\int_0^{2\pi} d\varphi\ \vec{i}_\rho(\varphi) = 0$

$$\vec{B}(x\vec{i}_x) = \frac{\mu_0 i}{4\pi}\int_0^{2\pi} d\varphi \, \frac{R^2 \vec{i}_x}{(R^2+x^2)^{3/2}}$$

$$\vec{B}(x\vec{i}_x) = \frac{\mu_0 i R^2}{2} \frac{\vec{i}_x}{(R^2+x^2)^{3/2}}$$

(b) $\vec{F} = q\,\vec{v}\times\vec{B}(x\vec{i}_x) = 0$

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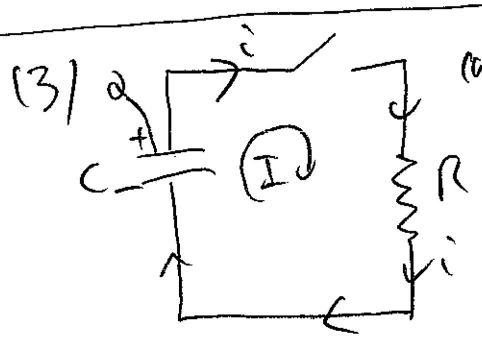
(3)



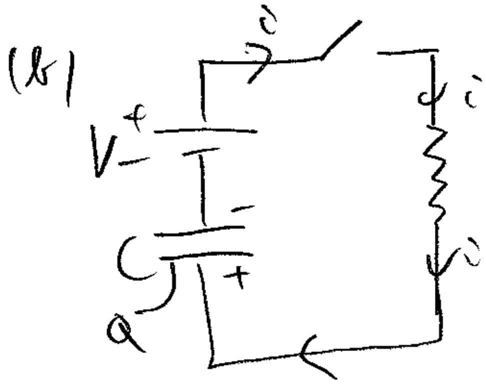
(a) Faraday's Law

$$\oint_C d\vec{r}\cdot\vec{E} = 0 \quad (L=0)$$

$$-\frac{Q}{C} + Ri = 0$$

$$i = -\dot{Q} \Rightarrow \dot{Q} = -\frac{Q}{RC} \Rightarrow \boxed{Q(t) = Q_0 \exp\left(-\frac{t}{RC}\right)}$$

(b)



$Q(0) = 0$

$$-V + \frac{Q}{C} + Ri = 0$$

$$\Rightarrow i + \frac{Q}{RC} = \frac{V}{R}$$

$$i = +\dot{Q}$$

$$\Rightarrow \dot{Q} + \frac{Q}{RC} = \frac{V}{R}$$

Homogeneous equation has general solution per above

$$Q_{hom} = A\exp\left(-\frac{t}{RC}\right) \text{ with } A = \text{const.}$$

Particular solution of inhom. eq.

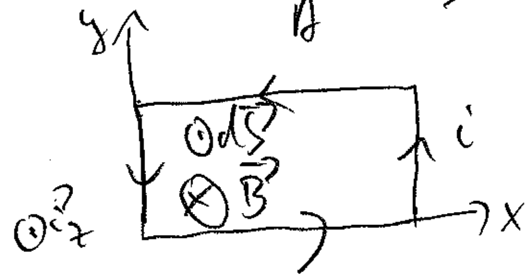
$$Q = A_1 = \text{const} \Rightarrow \frac{A_1}{RC} = \frac{V}{R} \Rightarrow A_1 = CV$$

$$Q(t) = CV + A \exp\left(-\frac{t}{RC}\right)$$

$$Q(0) = CV + A \overset{!}{=} 0 \Rightarrow A = -CV$$

$$\Rightarrow \boxed{Q(t) = CV\left[1 - \exp\left(-\frac{t}{RC}\right)\right]}$$

(4) $R = \frac{2(W+H)}{A}\rho \qquad \vec{B} = -B(t)\,\vec{e}_z$



Faraday's law

$$\oint_{\partial S} d\vec{r}\cdot\vec{E} = Ri = -\frac{d}{dt}(-HWB)$$

$$\Rightarrow i = \frac{HW}{R}\dot{B} = \frac{HWA}{2(W+H)\rho}\left(24t^3 + 7\right)$$