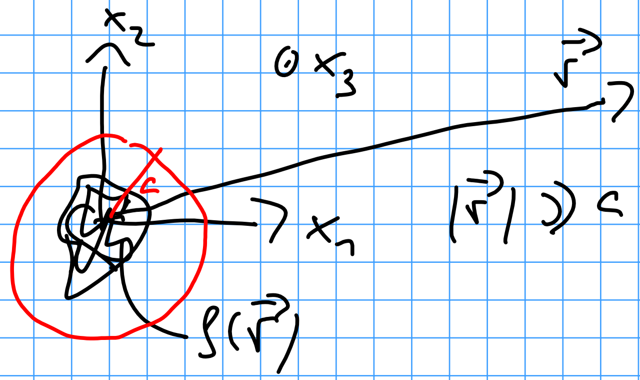
# Theo 2: Blatt 5

(2)




$$\Phi(\vec{r}) = \int_V d^3r' \, \frac{\rho(\vec{r}\,')}{4\pi\varepsilon_0 |\vec{r} - \vec{r}\,'|}$$

$$\vec{r}\,' \in V \subseteq K_a \quad ; \quad |\vec{r}\,'| \ll |\vec{r}|$$

(a) $$\frac{1}{|\vec{r} - \vec{r}\,'|} = \frac{1}{r} \cdot \frac{1}{\left|\frac{\vec{r}}{r} - \frac{\vec{r}\,'}{r}\right|}$$

$$\left|\frac{\vec{r}}{r} - \frac{\vec{r}\,'}{r}\right| = \frac{1}{r}|\vec{r} - \vec{r}\,'| = \frac{1}{r}\sqrt{(\vec{r} - \vec{r}\,')^2}$$

$$= \frac{1}{r}\left(r^2 + r'^2 - 2\vec{r}\cdot\vec{r}\,'\right)^{1/2}$$

$$= \left(1 + \left(\frac{r'}{r}\right)^2 - \frac{2\vec{r}\cdot\vec{r}\,'}{r^2}\right)^{1/2}$$

$$= (1 - \xi)^{1/2} \Rightarrow \xi = \underbrace{\frac{2\vec{r}\,\vec{r}\,'}{r^2}}_{O\left(\frac{r'}{r}\right)} - \underbrace{\left(\frac{r'}{r}\right)^2}_{O\left[\left(\frac{r'}{r}\right)^2\right]}$$

(b) $$\frac{1}{|\vec{r} - \vec{r}\,'|} = \frac{1}{r}(1-\xi)^{-1/2}$$

$$f(\xi) = (1-\xi)^{-1/2} = f(0) + \xi f'(0) + \frac{\xi^2}{2} f''(0) + O(\xi^3)$$

$$f(0) = 1$$

$$f'(\xi) = +\frac{1}{2}(1-\xi)^{-3/2} \Rightarrow f'(0) = \frac{1}{2}$$

$$f''(\xi) = +\frac{3}{4}(1-\xi)^{-5/2} \Rightarrow f''(0) = \frac{3}{4}$$

$$f(\xi) = (1-\xi)^{-1/2} = \frac{1}{\sqrt{1-\xi}} = 1 + \frac{\xi}{2} + \frac{3}{8}\xi^2 + O(\xi^3)$$

(c) $$\frac{1}{|\vec{r}-\vec{r}'|} = \frac{1}{r}\left(1 + \frac{\xi}{2} + \frac{3}{8}\xi^2 + O(\xi^3)\right)$$

$$= \frac{1}{r}\left[1 + \frac{\vec{r}\cdot\vec{r}'}{r^2} - \frac{1}{2}\left(\frac{r'}{r}\right)^2\right.$$

$$\left. \frac{3}{\cancel{8}\,2}\cdot\cancel{4}\,\frac{(\vec{r}\cdot\vec{r}')^2}{r^4}\right] + O\left(\frac{1}{r^5}\right)$$

$$\Phi(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int_{V_a} d^3r' \, \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|}$$

$$= \frac{1}{4\pi\varepsilon_0 r}\int_{V_a} d^3r' \left[1 + \frac{\vec{r}\cdot\vec{r}'}{r^2} + \frac{3(\vec{r}\cdot\vec{r}')^2 - r'^2 r^2}{2r^4}\right] \rho(\vec{r}')$$

$$= \frac{Q}{4\pi\varepsilon_0 r} + \frac{\vec{p}\cdot\vec{r}}{4\pi\varepsilon_0 r^3} + \frac{1}{4\pi\varepsilon_0}\cdot\frac{q_{jk}x_j x_k}{2r^5}$$

$\vec{p} = \int_{V_a} d^3r' \, \vec{r}' \rho(\vec{r}')$ : Dipolmoment der Ladungsverteilung

$\oplus\, Q$, $d\,\uparrow \vec{p}$, $\ominus\, -Q$  $d \to 0$ ; $Qd = \text{const.}$

$$\int_{V_a} d^3r'\, \rho(\vec{r}\,')\left[3(\vec{r}\cdot\vec{r}\,')^2 - r'^2 r^2\right]$$

$$= \int_{V_a} d^3r'\, \rho(\vec{r}\,')\left[3 x_j x_k x_j' x_k' - r'^2 \delta_{jk} x_j x_k\right]$$

$$= x_j x_k \underbrace{\int_{V_a} d^3r'\, \rho(r')\left[3x_j' x_k' - r'^2 \delta_{jk}\right]}_{q_{jk}}$$

$$\Phi(\vec{r}) = \underbrace{\frac{Q}{4\pi\varepsilon_0 r}}_{\mathcal{O}\left(\frac{1}{r}\right)} + \underbrace{\frac{\vec{p}\cdot\vec{r}}{4\pi\varepsilon_0 r^3}}_{\mathcal{O}\left(\frac{1}{r^2}\right)} + \underbrace{\frac{q_{jk} x_j x_k}{8\pi\varepsilon_0 r^5}}_{\mathcal{O}\left(\frac{1}{r^3}\right)} + \mathcal{O}\left(\frac{1}{r^4}\right)$$

Monopol  Dipol  Quadrupol

$\ominus\ \oplus$ / $\oplus\ \ominus$ Quadrupol  $q_{jk} = q_{kj}$ ; $q_{jj} = 0$

5 unabhängige Einträge in Matrix $q_{kj}$

(d) $\vec{E} = -\vec{\nabla}\Phi$ ; $E_i = -\partial_i \Phi$

$$\partial_j\left(\frac{1}{r}\right) = -\frac{1}{r^2}\frac{x_j}{r} = -\frac{x_j}{r^3} ;$$

$$\frac{\partial}{\partial x_j} r = \frac{\partial}{\partial x_j}\sqrt{x_1^2 + x_2^2 + x_2^2} = \frac{1}{2r} 2x_j = \frac{x_j}{r}$$

$$\vec{\nabla} r = \frac{\vec{r}}{r} = \hat{r}$$

$$\vec{E}_{\text{monopol}} = \frac{Q}{4\pi\varepsilon_0} \cdot \frac{\vec{r}}{r^3} \qquad \text{(Coulomb-Feld)}$$

$$\vec{E}_{\text{Dipol}} = -\vec{\nabla} \left( \frac{\vec{p}\cdot\vec{r}}{4\pi\varepsilon_0 r^3} \right)$$

$$= -\frac{\vec{p}}{4\pi\varepsilon_0 r^3} + \frac{\vec{p}\cdot\vec{r}}{4\pi\varepsilon_0} \left( + \frac{3}{r^4} \frac{\vec{r}}{r} \right)$$

$$\vec{E}_{\text{Dipol}} = \frac{3(\vec{p}\cdot\vec{r})\,\vec{r} - r^2 \vec{p}}{4\pi\varepsilon_0 r^5}$$

$$E_j^{(\text{Quad})} = -\partial_j \frac{q_{kl}\, x_k x_l}{8\pi\varepsilon_0 r^5}$$

$$= \frac{-1}{8\pi\varepsilon_0} \left[ \frac{q_{kl}}{r^5} \left( \delta_{jk} x_l + \delta_{jl} x_k \right) + q_{kl}\, x_k x_l \left( -\frac{5}{r^6} \cdot \frac{x_j}{r} \right) \right]$$

$$= -\frac{1}{8\pi\varepsilon_0 r^7} \left[ r^2 q_{jl} x_l + r^2 \overbrace{q_{kj}}^{q_{jk}} x_k - 5\, q_{kl}\, x_k x_l x_j \right]$$

$$= -\frac{1}{8\pi\varepsilon_0 r^7} \left[ 2 r^2 q_{jk} x_k - 5\, q_{kl}\, x_k x_l x_j \right]$$

$$E_j^{(\text{Quad})} = \frac{5\, q_{kl}\, x_k x_l x_j - 2 r^2 q_{jk} x_k}{8\pi\varepsilon_0 r^7}$$