Phase diagram for QCD
Phase diagram for QCD

\[ \begin{align*}
T \\
\mu
\end{align*} \]
Phase diagram for QCD

- Hadrons
- Nuclear matter
- Critical point?
- Quark-Gluon Plasma
- Colour Superconductor?
Overview on Lattice QCD

- Discretize Euclidean space-time by a hypercubic lattice $\Lambda$
- Quantize QCD using Euclidean path integrals
- Calculate expectation values using Monte Carlo techniques:

$$\langle A \rangle = \frac{1}{Z} \int \mathcal{D}[U] A[U] \ (\det D)^{N_f} \ e^{-S_G[U]} \quad U_{x,\mu} \in SU(3)$$

- **Ab initio** method: Only QCD parameters $m_q \rightarrow \kappa$, $g_0 \rightarrow \beta$. 
The SIGN problem

- Finite chemical potential $\rightarrow$ SIGN problem

$$(\text{det } D(\mu))^* = \text{det } D(-\mu^*) \rightarrow \text{det } D(\mu \neq 0) \in \mathbb{C}.$$ 

- Importance Sampling based Monte Carlo methods fail

$$\langle O \rangle = \frac{1}{Z} \int \mathcal{D}[U] \; O(U) \; |\text{det } D| \; e^{i\phi} \; e^{-S_G(U)}$$
How bad is the SIGN problem?!

- Overlap problem

\[
\langle O \rangle = \frac{\int \mathcal{D}[U] \ O \ |\det D| \ e^{i\phi} \ e^{-S_G(U)}}{\int \mathcal{D}[U] \ |\det D| \ e^{i\phi} \ e^{-S_G(U)}} = \frac{\langle O \ e^{i\phi} \rangle_{pq}}{\langle e^{i\phi} \rangle_{pq}}
\]

- The average phase \textit{vanishes} in the thermodynamic limit: \( V \to \infty \)

\[
\langle e^{i\phi} \rangle_{pq} = e^{-V \cdot \Delta F} \to 0
\]
Alternatives?

**Taylor expansion**
- Expand around small $\mu$

$$\frac{T_c(\mu)}{T_c(0)} = 1 + c_2 \cdot \left( \frac{\mu}{T_c(0)} \right)^2 + c_4 \cdot \left( \frac{\mu}{T_c(0)} \right)^4 + \ldots$$

- Radius of convergence?

**Reweighting**
- Absorb phase in the observable

$$\langle O \rangle = \frac{\int \mathcal{D}[U] \ O \ |\det D| \ e^{i\phi} \ e^{-S_G(U)}}{\int \mathcal{D}[U] \ |\det D| \ e^{i\phi} \ e^{-S_G(U)}} = \frac{\langle O \ e^{i\phi} \rangle_{pq}}{\langle e^{i\phi} \rangle_{pq}}$$

- Overlap problem
Alternatives?

**Analytic continuation**

- Continue from imaginary $\mu$ to real $\mu$

- How far can one extrapolate?

**Many more ...**

**Here: Complex Langevin**

- Stochastic simulation with complexified degrees of freedom
Complex Langevin - Idea

• Consider a simple Gaussian integral

\[ Z(a, b) = \int dx \, e^{-S}, \text{ with } S = \frac{1}{2}ax^2 + ibx \]

• Solution: Complete the square or complexify \( x \rightarrow z = x + iy \)

• Consider **Real action**

• Langevin equation: Stochastic process (Brownian motion)

\[ \frac{\partial x}{\partial t} = \frac{\partial S}{\partial x} + \eta(t) \]

where \( \langle \eta(t)\eta(t') \rangle = 2\delta(t - t') \) Gaussian white noise

• The associated Fokker-Plank equation

\[ \langle O(x) \rangle_{\eta} = \int dx \, \rho(x, t) \, O(x) \]

• Stationary solution \( \rho(x) = e^{-S} \)
Complex Langevin - Idea

• Consider **Complex action**
• Complexify \( x \rightarrow z = x + iy \) Langevin equation

\[
\frac{\partial x}{\partial t} = \text{Re} \frac{\partial S}{\partial z} + \eta(t) \quad \text{and} \quad \frac{\partial y}{\partial t} = \text{Im} \frac{\partial S}{\partial z}
\]

• The associated Fokker-Plank equation

\[
\langle O(x + iy) \rangle_\eta = \int dx\, dy \, \tilde{\rho}(x + iy) \, O(x + iy)
\]

• Convergence:
  • Imaginary direction \( y \) is compact (No boundary terms)
  • Action \( S \) is holomorphic

\[
\Rightarrow \int dx\, dy \, \tilde{\rho}(x + iy) \, O(x + iy) = \int dx \, \rho(x, t) \, O(x)
\]
Complex Langevin - Gauge theories

- Complexify degrees of freedom $SU(3) \rightarrow SL(3, \mathbb{C})$

$$U_{x,\mu} = \exp \left[ i \alpha \lambda^c \left( A^c_{x,\mu} + i B^c_{x,\mu} \right) \right]$$

- Evolve links according (1st order) Langevin equation

$$U_{x,\mu}(\theta + \varepsilon) = \exp \left[ i \lambda^a \left( -\varepsilon D^a_{x,\mu} S + \sqrt{\varepsilon} \eta^a_{x,\mu} \right) \right] U_{x,\mu}(\theta)$$
Complex Langevin simulations

- However, $\text{SL}(3, \mathbb{C})$ is not a compact group...
- Convergence $\Leftrightarrow$
  - Action $S$ is holomorphic
  - "Imaginary" direction of $\text{SL}(3, \mathbb{C})$ falls off quickly enough
- Measure distance to $\text{SU}(3)$ manifold

$$\text{unitnorm} = \text{Tr} \left( U_{x,\mu} U_{x,\mu}^\dagger - 1 \right)^2$$

- Gauge cooling is essential, but sometimes not sufficient...

$$U_{x,\mu} \rightarrow \Omega_x \ U_{x,\mu} \ \Omega_{x+\mu}^{-1}$$
Gauge cooling

HDQCD: $10^3 \times 4, \mu = 0.7, \beta = 5.8, \kappa = 0.04, N_f = 2$

- Tunneling to wrong results.
Gauge cooling

HDQCD: $10^3 \times 4, \mu = 0.7, \beta = 5.8, \kappa = 0.04, N_f = 2$

- Tunneling to wrong results, when unitnorm grows too large.
Dynamic stabilization

- Adding a trivial force to the Langevin dynamics

\[ U_{x,\nu}(\theta + \varepsilon) = \exp \left[ i \lambda^a \left( \varepsilon K^a_{x,\nu} + i \varepsilon \alpha_{DS} M^a_x + \sqrt{\varepsilon} \eta^a_{x,\nu} \right) \right] U_{x,\nu}(\theta) \]

where

\[ M^a_x = i b^a_x \left( \sum_c b^c_x b^c_x \right)^3 \text{ and } b^a_x = \text{Tr} \left[ \lambda^a \sum_{\nu} U_{x,\nu} U_{x,\nu}^\dagger \right]. \]

- Expanding the force in terms of gauge fields \( A \) and \( B \)

\[ M^a_x \sim a^7 \left( \overline{B}^c_y \overline{B}^c_y \right)^3 \overline{B}^a_x + \mathcal{O}(a^8). \]

- Dynamic stabilization is numerically cheap and can be combined with gauge cooling (Here: 1 step)
Dynamic stabilization

For sufficient large $\alpha_{DS}$ we find agreement with reweighting.
Dynamic stabilization

HDQCD: 10^3 × 4, µ = 0.7, β = 5.8, κ = 0.04, N_f = 2

- Improved stability using dynamic stabilization
Results

- **Pure Yang-Mills** ⇔ No sign problem
  - Checking Langevin simulations
  - Compare to standard Hybrid Monte Carlo methods

- **Heavy dense QCD (HDQCD)**
  - Heavy and dense approximation of QCD
  - Has a sign problem and phase transitions
  - Numerical cheap :)

- **XY model with finite $\mu$**
  - Toy model for QCD

- **Full (Staggered) QCD**
  - Including light dynamical fermions
  - Very preliminary results
The correct transition is obtained, even for SL(3, C) start.
The correct transition is obtained, even for $SL(3, \mathbb{C})$ start.
Heavy Dense QCD

- **Here: QCD in the limit of heavy quarks (HDQCD).**
  - Fermion determinant simplifies with the Polyakov loops $P_\vec{x}$ and $P_{\vec{x}}^{-1}$ as
  \[
  \det D(\mu) = \prod_{\vec{x}} \det (1 + C P_{\vec{x}})^2 \det (1 + C' P_{\vec{x}}^{-1})^2,
  \]
  where the Polyakov loop $P_{\vec{x}}$ is defined as
  \[
  P_{\vec{x}} = \frac{1}{V} \sum_t U_0(\vec{x}, t)
  \]
  - For the gluonic part, we use the full Wilson gauge action.
  - Map out the phase diagram for HDQCD.
  - Expected transition: $\mu_c^0 = -\ln(2\kappa)$
Strategy:

- Determine $\mu$-transition in Fermion density
- Determine $T$-transition in Polyakov loop
Heavy Dense QCD

\[ \beta = 5.8 \]
\[ \kappa = 0.04 \]
\[ V = 6^3, 8^3, 10^3 \]
\[ N_f = 2 \]
\[ a \sim 0.15 \text{ fm} \]
\[ \mu_c^0 = 2.53 \]
\[ N_{\tau} \]
\[ T [\text{MeV}] \]
\[ 28 - 2 \]
\[ 48 - 671 \]
Fermion density: $n = \frac{1}{N_t} \frac{\partial \ln Z}{\partial \mu}$
Heavy Dense QCD

• Polyakov loop

Benjamin Jäger  Nuclear Physics Kolloqium  05.07.2017
Binder cumulant of the Polyakov loop $B = 1 - \frac{\langle P^4 \rangle}{3 \langle P^2 \rangle^2}$
Fit the phase boundary using \( T_c(\mu) = \sum_k b_k (1 - \mu/\mu_c)^k \)
XY model with finite $\mu$

- Toy model to test Complex Langevin
- Has a phase transition at $\beta_c \sim 0.45$
- Ordered phase: $\beta > \beta_c$, disordered phase: $\beta < \beta_c$

\[
S = -\beta \sum_x \sum_{\nu=0}^{2} \cos(\phi_x - \phi_{x+\nu} - i\mu\delta_{\nu,0})
\]
In the ordered phase Complex Langevin agrees with (dual) Worldline formulation.
In the disordered phase Complex Langevin fails spectacularly.
**XY model with finite \( \mu \)**

\[ \text{XY model } \beta = 0.2 \]

\[ S/\Omega \]

\[ \mu^2 \]

- **Real**
- **DS, \( \alpha = 10^5 \)**
- **Without DS**
- **Worldline**

- Adding dynamic stabilization
XY model with finite $\mu$

For large $\mu^2$ a discrepancy remains
Naïve Staggered QCD

- Here: QCD using unimproved Staggered quarks.
  - Setup: $\beta = 5.6, a m_q = 0.025$
  - Every Langevin update needs inversion of the Dirac operator . . .

\[ D_a^{x,\mu} S = D_a^{x,\mu} S_{YM} - \frac{N_f}{4} \text{Tr} \left[ M^{-1} D_a^{x,\mu} M \right] \]

- Comparison @ $\mu = 0$ with HMC
Naïve Staggered QCD

- Comparison @ $\mu = 0$ with HMC: $6^4$
Naïve Staggered QCD

- Comparison @ $\mu = 0$ with HMC: $6^4$ - Stepsize extrapolation
Naïve Staggered QCD

- Comparison @ $\mu = 0$ with HMC: $6^4$ - Stepsize extrapolation
Naïve Staggered QCD

- Comparison @ $\mu = 0$ with HMC: $8^4$ - Stepsize extrapolation
Naïve Staggered QCD

- Comparison @ $\mu = 0$ with HMC: $8^4$ - Stepsize extrapolation
Naïve Staggered QCD

- **Comparison @ \( \mu = 0 \) with HMC.**

<table>
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<td>CL</td>
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<tr>
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<td>0.1372(3)</td>
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<td>( 12^4 )</td>
<td>0.58196(6)</td>
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</tr>
</tbody>
</table>
Naïve Staggered QCD

- Fermion density
Naïve Staggered QCD

- Polyakov loop

\[ \mu/T \]

\( N_\tau = 2 \)
\( N_\tau = 4 \)
\( N_\tau = 6 \)
\( N_\tau = 8 \)
Future work

Conclusion

- Complex Langevin simulation can be used to study the QCD phase diagram
- Dynamical stabilisation improves convergence
- Work on the convergence, especially around $\mu_c$.

Future work

- Start proper Full QCD simulations to identify phase structure of QCD.
- A lot of work to be done!
Thank you for your attention!