# On Energy-Momentum & Spin: the inertial currents in classical field theory

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Hehl, McCrea, Mielke, Ne'eman, Phys. Reports (1995), Y.N. Obukhov, T. Ramos & G. Rubilar, Phys. Rev. E86 (2012) 031703, some slides are adapted from Yuri Obukhov

*file FrankfurtIAS03.tex*

F. Becattini and L. Tinti, *Thermodynamical inequivalence of quantum stress-energy and spin tensors,* Phys. Rev. D **84**, 025013 (2011) [arXiv:1101.5251].

F. Becattini and L. Tinti, *Nonequilibrium Thermodynamical Inequivalence of Quantum Stress-energy and Spin Tensors,* Phys. Rev. D **87**, 025029 (2013) [arXiv:1209.6212].

"It is shown that different pairs of stress-energy and spin tensors of quantum relativistic fields related by a pseudo-gauge transformation, i.e., differing by a divergence, imply different mean values of physical quantities in thermodynamical nonequilibrium situations. Most notably, transport coefficients and the total entropy production rate are affected by the choice of the spin tensor of the relativistic quantum field theory under consideration. Therefore, at least in principle, it should be possible to disprove a fundamental stress-energy tensor and/or to show that a fundamental spin tensor exists by means of a dissipative thermodynamical experiment."

On Energy-Momentum & Spin: the inertial currents in classical field theory

1. Action principle, translational invariance
2. Lorentz invariance
3. Poincaré invariance
4. On formalism, the electromagnetic energy-momentum, the Dirac field, and on open and closed systems
5. Relocalization of energy-momentum and spin
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7. Dynamic Sciama-Kibble spin in Poincaré gauge theory

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[9. An algebra of the momentum and the spin currents? ]

# 1. Action principle, translational invariance

- SR, Minkowski spacetime $M_4$, Lorentz metric $g_{ij} \stackrel{*}{=} o_{ij} := \mathrm{diag}(+---)$, coordinates $x^i$, $i, j, k, ... = 0, 1, 2, 3$; here Cartesian coo., matter field $\Psi$, could be a scalar, Weyl, Dirac, Maxwell, Proca, Rarita-Schwinger, Fierz-Pauli field etc.). Isolated material system with 1st order action (see Landau-Lifshitz, Corson): $W_{\text{mat}} := \frac{1}{c} \int d\Omega \mathcal{L}(\Psi, \partial\Psi)$.
- Invariance under 4 transl.: $x'^i = x^i + a^i$. Noether theorem and $\frac{\delta\mathcal{L}}{\delta\Psi} = 0$,

$$\boxed{\partial_j \mathfrak{T}_i{}^j = 0}\,, \qquad \underbrace{\mathfrak{T}_i{}^j}_{4\times 4} := \mathcal{L}\delta_i^j - \frac{\partial\mathcal{L}}{\partial\partial_j\Psi}\partial_i\Psi$$

  canonical energy-momentum tensor of type $\binom{1}{1}$, Noether energy-momentum (or momentum current density), 16 indep. comps., Whittaker: Minkowski's most important discovery; is asymmetric a priori
- Physical components of components of $\mathfrak{T}_i{}^j$ (a,b=1,2,3):

$$\mathfrak{T} \quad = \begin{pmatrix} \mathfrak{T}_0{}^0 = -\text{energy d.} & \mathfrak{T}_0{}^b = (\text{energy flux d.}) \times c \\ \mathfrak{T}_a{}^0 = -(\text{mom. d.})/c & \mathfrak{T}_a{}^b = -\text{mom. flux d.} \end{pmatrix}$$

$$\left[\mathfrak{T}_i{}^j\right] \quad = \underbrace{h\,\ell^{-3}t^{-1}}_{\text{stress}} \times \begin{pmatrix} 1 & \ell/t \\ (\ell/t)^{-1} & 1 \end{pmatrix}$$

  here $h :=$ [action], $\ell :=$ [length], $t :=$ [time], method of Dorlego-Schouten. Lagrangian: $[\mathcal{L}] = \frac{h}{\ell^3 t}$, $\frac{h}{t} =$ [energy].

- Note, the spatial components $[\mathfrak{T}_a{}^b] = (mv)v\,\frac{1}{\ell^3} = \frac{E}{\ell^3} = \frac{f}{\ell^2}$ = stress, see Lorentz's interpretation of the Maxwell stress.
- Semiclassical Weyssenhoff ansatz for a fluid:

$$\underbrace{\mathfrak{T}_i{}^j}_{\text{mom. curr. d.}} = \underbrace{\mathfrak{p}_i}_{\text{mom. d.}}\ \underbrace{v^j}_{\text{velocity}}\,, \qquad \text{observe natural index positions!}$$

- If $\mathfrak{p}_i = \underbrace{\rho}_{\text{mass d.}}\, g_{ik} v^k$, then $\mathfrak{T}_{ij} = \mathfrak{T}_{ji}$ symm.. Is *not* the case for spin fluids
- Superfluid $^3$He in the A-phase, is as spin fluid (Lee, Osheroff, Richardson 1972). Take the *angular momentum law*, see D. Vollhardt and P. Wölfle, *The Superfluid Phases of Helium 3*, London 1990, p.427: The antisym. piece of stress reads:

$$\underbrace{\epsilon_{ijk}\Pi_{jk}}_{\sim\epsilon_{abc}\mathfrak{T}^{bc}} = -\left(\frac{\partial}{\partial t} + \mathbf{v}_{\rm n}\cdot\boldsymbol{\nabla}\right)(t_0 l_i) + \nabla_j B_{ji} - \underbrace{\nabla_j}_{\sim\nabla^b}\{\underbrace{\frac{\hbar}{2m} g_{s,j}\, l_i}_{\sim\mathfrak{p}_b\epsilon_{acd}\mathfrak{s}^{cd}} + [\hat{\mathbf{l}}\times T\,\frac{\partial s}{\partial(\nabla_j\hat{\mathbf{l}})}]_i\}$$

$v_{\rm n}$ = velocity of normal fluid, $t_0$ = modulus of intrinsic angular momentum $\mathbf{t} = t_0\hat{\mathbf{l}}$, $l_i$ = preferred direction of A-phase order parameter, $s$ = entropy density, $T$ = temperature, $g_{\rm s}$ = momentum density of superfluid component; this is an irrefutable proof that asymmetric stress tensors exist in nature (see Pascal-Euler-Cauchy-Boltzmann-Voigt-the Cosserats-E.Cartan...)

- In a spacetime with metric, as in the $M_4$, we can decompose $\mathfrak{T}_{ij}$ irreducible wrt the Lorentz group:

$$\begin{aligned} \mathfrak{T}_{ij} &= \quad \not{\mathfrak{T}}_{ij} \quad + \quad \mathfrak{T}_{[ij]} \quad + \quad \frac{1}{4} g_{ij} \mathfrak{T}_k{}^k \\ 16 &= 9\,(\text{sym.tracefree}) \oplus 6(\text{antisym.}) \oplus 1\,(\text{trace})\,, \end{aligned}$$

$\not{\mathfrak{T}}_{ij} := \mathfrak{T}_{(ij)} - \frac{1}{4} g_{ij} \mathfrak{T}_k{}^k$, Bach parentheses $(ij) := \frac{1}{2}\{i+j\}$, $[ij] := \frac{1}{2}\{i-j\}$.

- In electromagnetism, only $\not{\mathfrak{T}}_{ij}$ survives (9 components), since it is massless, that is, $\mathfrak{T}_k{}^k = 0$, and carries *helicity*, but no (Lorentz) spin, i.e., $\mathfrak{T}_{[ij]} = 0$, see below.
- Classical ideal (perfect, Euler) fluid of GR ($\rho$ = mass/energy density, $p$ = pressure, $u_i$ = velocity of fluid):

$$\mathfrak{T}_{ij} = (\rho + p) u_i u_j - p g_{ij}\,, \qquad \mathfrak{T}_{[ij]} = 0\,, \qquad \mathfrak{T}_k{}^k = \rho - 3p\,.$$

- Where took Einstein the symmetry of the energy-momentum tensor from? Einstein (*The Meaning of Relativity,* 1922, p.50) discussed the symmetry of the energy-momentum tensor of Maxwell's theory. Subsequently, he argued: "*We can hardly avoid making the assumption that in all other cases, also, the space distribution of energy is given by a symmetrical tensor, $T_{\mu\nu}$, ...*" This is hardly a convincing argument if one recalls that the Maxwell field is massless.

## 2. Lorentz invariance

- Invariance under 3+3 Lorentz transf.: $x'^i = x^i + \omega^{ij} x_j$, with $\omega^{(ij)} = 0$ Noether theorem and $\frac{\delta \mathcal{L}}{\delta \Psi} = 0$,

$$\partial_k \big( \underbrace{\mathfrak{S}_{ij}{}^k}_{\text{spin}} + \underbrace{\frac{1}{2} x_i \mathfrak{T}_j{}^k - \frac{1}{2} x_j \mathfrak{T}_i{}^k}_{\text{orbital angular momentum}} \big) = 0\,, \qquad \underbrace{\mathfrak{S}_{ij}{}^k}_{6\times 4} := -\frac{\partial \mathcal{L}}{\partial \partial_k \Psi} f_{ij} \Psi$$

canon. or Noether spin $\mathfrak{S}_{ij}{}^k = -\mathfrak{S}_{ji}{}^k$, the spin current density, is a tensor of type $\binom{1}{2}$, see also Einstein-de Haas effect (1915) .

- Physical components of $\mathfrak{S}_{ij}{}^k$ (a,b=1,2,3):

$$\mathfrak{S} \quad = \begin{pmatrix} \mathfrak{S}_0{}^{b0} = \text{en.-dipole mom. d.} & \mathfrak{S}_0{}^{bc} = (\text{en.-dipole mom. flux d.})/c \\ \mathfrak{S}_a{}^{b0} = (\text{spin density}) \times c & \mathfrak{S}_a{}^{bc} = \text{spin flux density} \end{pmatrix}$$

$$\left[ \mathfrak{S}_i{}^{jk} \right] \quad = \underbrace{h\, \ell^{-2} t^{-1}}_{\text{moment stress}} \times \begin{pmatrix} 1 & \ell/t \\ (\ell/t)^{-1} & 1 \end{pmatrix}$$

- $[\mathfrak{S}_a{}^{bc}] = (mv\ell)v\frac{1}{\ell^3} = \frac{f\ell}{\ell^2}$ = moment stress, known from Voigt (1887) and from the Cosserat brothers (1909), from micropolar media,...

- Convective Weyssenhoff ansatz (distinguish spin current from spin):

$$\underbrace{\mathfrak{S}_{ij}{}^{k}}_{\text{spin curr. d.}} = \underbrace{\mathfrak{s}_{ij}}_{\text{spin}} \underbrace{v^{k}}_{\text{velocity}} = -\mathfrak{S}_{ji}{}^{k}$$

- Irreducible decomposition:

$$\begin{aligned} \mathfrak{S}_{ij}{}^{k} &= {}^{\text{TEN}}\mathfrak{S}_{ij}{}^{k} + {}^{\text{TRA}}\mathfrak{S}_{ij}{}^{k} + {}^{\text{AX}}\mathfrak{S}_{ij}{}^{k} \\ 24 &= 16 \oplus 4 \oplus 4 \end{aligned}$$

$$\text{with} \quad {}^{\text{AX}}\mathfrak{S}_{ijk} := \mathfrak{S}_{[ijk]} \quad \text{and} \quad {}^{\text{TRA}}\mathfrak{S}_{ij}{}^{k} := \frac{2}{3}\mathfrak{S}_{[i|\ell}{}^{\ell}\delta^{k}_{|j]}$$

For the Dirac field we will find out

$$\overset{D}{\mathfrak{S}}_{ijk} = \overset{D}{\mathfrak{S}}_{[ijk]}\,, \qquad \text{that is, only} \qquad {}^{\text{AX}}\overset{D}{\mathfrak{S}}_{ijk} \neq 0 \quad \text{(4 components)}$$

- Back to the angular momentum law. Differentiate and apply $\partial_k \mathfrak{T}_i{}^k = 0$:

$$\partial_k \left( \mathfrak{S}^{ijk} + x^{[i}\mathfrak{T}^{j]k} \right) = 0 \qquad \Longrightarrow \qquad \boxed{\partial_k \mathfrak{S}^{ijk} - \mathfrak{T}^{[ij]} = 0}$$

The boxed version can be generalized to Riemann(-Cartan) spacetimes directly, see below. If $\mathfrak{S}^{ijk} = 0$, then $\mathfrak{T}^{[ij]} = 0$ (symmetric energy-momentum tensor), but not necessarily vice versa.

# 3. Poincaré invariance

- Thus, Poincaré invariance yields the $4+6$ conservation laws

$$\begin{aligned} \partial_k \mathfrak{T}_i{}^k &= 0 && \text{(energy-momentum)} \\ \partial_k \mathfrak{S}_{ij}{}^k - \mathfrak{T}_{[ij]} &= 0 && \text{(angular momentum)} \end{aligned}$$

The angular momentum law reflects the semi-direct product structure of the Poincaré group. Recall its Lie algebra:

$$\begin{aligned} [P_i, P_j] &= 0\,, \\ [L_{ij}, P_k] &= g_{k[i} P_{j]}\,, \quad \text{(transl. and Lorentz transf. mix, see } \mathfrak{S}_{ijk} + x_{[i}\mathfrak{T}_{j]k}) \\ [L_{ij}, L_{k\ell}] &= g_{k[i} L_{j]\ell} - g_{\ell[i} L_{j]k}\,. \end{aligned}$$

- The rigid Poincaré group of SR can be gauged [see Blagojević & H.(eds.) *Gauge Theories of Gravitation* (2013)] yielding a Riemann-Cartan spacetime. Then, in particular, the conserv. laws generalize to

$$\begin{aligned} \overset{*}{\nabla}_k \, \mathfrak{T}_i{}^k &= \overbrace{C_{ik}{}^{\ell}}^{\text{torsion}} \, \mathfrak{T}_\ell{}^k + \overbrace{R_{ik}{}^{lm}}^{\text{curvature}} \, \mathfrak{S}_{lm}{}^k\,, \\ \overset{*}{\nabla}_k \, \mathfrak{S}_{ij}{}^k - \mathfrak{T}_{[ij]} &= 0\,. \end{aligned}$$

Here $\overset{*}{\nabla}_k := \nabla_k + C_{k\ell}{}^{\ell}$. General relativity is the subcase for $\mathfrak{S}_{ij}{}^k = 0$. Otherwise, the Einstein-Cartan(-Sciama-Kibble) theory with $C_{ij}{}^k \neq 0$. In GR and in EC the *Noether* theorems for translation + Lorentz can be mapped to the *Bianchi* identities.

## 4. On formalism, the electromagnetic energy-momentum, the Dirac field, and on open and closed systems

- Here it would be time in introduce the calculus of *exterior differential forms* in order to streamline the Lagrange-Noether formalism. In such a formalism one works with an orthonormal coframe (tetrad) $\vartheta^\alpha = e_i{}^\alpha dx^i$, a Lorentz connection $\Gamma^{\alpha\beta} = \Gamma_i{}^{\alpha\beta} dx^i = -\Gamma^{\beta\alpha}$, and the fields are exterior forms (0-forms, 1-forms,..., 4-forms) with values in the algbra of some Lie group. The electromagnetic potential is a 1-form $A = A_i dx^i$, the field strength a 2-form $F := dA = \frac{1}{2} F_{ij} dx^i \wedge dx^j$, for details see Hehl & Y.N. Obukhov, *Foundations of Electrodynamics:* Charge, flux, and metric, Birkhäuser, Boston (2003).
- We will not use this formalism heavily, but here only quote two interesting result: In exterior calculus, one works with the geometric objects and *not* with the *components* therefrom. For Maxwell's vacuum field, the potential $A$ , this has the consequence that the canonical (i.e. Noether) energy-momentum tensor is symmetric and gauge invariant directly, *without symmetrization,* that is, without the gauge-dependent artifacts created in the component formalism (à la Landau-Lifshitz).

- The second example, Dirac field in exterior calculus for illustration:

$$L_{\rm D} = \frac{i}{2}(\overline{\Psi}{}^\star\gamma \wedge D\Psi + \overline{D\Psi} \wedge {}^\star\gamma\Psi) + {}^\star m\overline{\Psi}\Psi$$

  with $\gamma := \gamma_\alpha \vartheta^\alpha$ and $\gamma_{(\alpha}\gamma_{\beta)} = o_{\alpha\beta}\mathbf{1}_4$. The 3-forms of the canonical momentum and spin current densities ($D_\alpha := e_\alpha \rfloor D$):

$$\begin{aligned} \mathfrak{T}_\alpha &= \frac{i}{2}\left(\overline{\Psi}\,{}^\star\gamma \wedge D_\alpha\Psi + \overline{D_\alpha\Psi} \wedge {}^\star\gamma\Psi\right), \\ \mathfrak{S}_{\alpha\beta} &= \frac{1}{4}\vartheta_\alpha \wedge \vartheta_\beta \wedge \overline{\Psi}\gamma\gamma_5\Psi\,. \end{aligned}$$

  In Ricci calculus $\mathfrak{S}_{\alpha\beta\gamma} = \mathfrak{S}_{[\alpha\beta\gamma]} = \frac{1}{4}\epsilon_{\alpha\beta\gamma\delta}\overline{\Psi}\gamma_5\gamma^\delta\Psi$ and $\mathfrak{t}_{\alpha\beta} = \mathfrak{T}_{(\alpha\beta)}$ (Tetrode), see subsequent slide. These are the *inertial currents* (and thus the gravitational currents) of the classical Dirac field. A decomposition of $(\mathfrak{T}_\alpha, \mathfrak{S}_{\alpha\beta})$ à la Gordon, yields the *gravitational* moment densities of the Dirac field (`arXiv:gr-qc/9706009`); is a special case of relocalization, see below.

- Off shell, the Noether theorems read:

$$\partial_k \mathfrak{T}_i{}^k \equiv -\frac{\delta\mathcal{L}}{\delta\Psi^A}\partial_i\Psi^A, \qquad \partial_k\mathfrak{S}_{ij}{}^k - \mathfrak{T}_{[ij]} \equiv -\frac{\delta\mathcal{L}}{\delta\Psi^A}(f_{ij})^A{}_B\Psi^B\,.$$

  If we have external fields, we separate fields in two subsets $\Psi^A = \left\{\Psi^{\mathcal{A}}_{\rm dyn}, \Psi^\alpha_{\rm ext}\right\}$. For external fields we have, in general, $\delta\mathcal{L}/\delta\Psi^\alpha_{\rm ext} \neq 0$.

# 5. Relocalization of energy-momentum and spin

- Canonical currents are not uniquely defined. *Relocalization*

$$\widehat{\mathfrak{T}}_i{}^j = \mathfrak{T}_i{}^j - \partial_l X_i{}^{jl} ,$$
$$\widehat{\mathfrak{S}}_{kl}{}^j = \mathfrak{S}_{kl}{}^j - 2X_{[kl]}{}^j + \partial_i Y_{kl}{}^{ji} .$$

Still,
$$\partial_j \widehat{\mathfrak{T}}_i{}^j = 0 ,$$
$$\partial_j \widehat{\mathfrak{S}}_{kl}{}^j - \widehat{\mathfrak{T}}_{[kl]} = 0 .$$

Arbit. $X_i{}^{jl} = -X_i{}^{lj}$, $Y_{kl}{}^{ji} = -Y_{kl}{}^{ij} = -Y_{lk}{}^{ji}$ (Hehl, Rep.Math.Phys.1976). Integrated total energy-mom. and angular momentum remain the same.

- *Belinfante* relocalization (1939): Require $\widehat{\mathfrak{S}}_{kl}{}^j = 0$. Resolve wrt $X$. Then,

$$X_i{}^{jl} = -\frac{1}{2}\left(\mathfrak{S}^{jl}{}_i + \mathfrak{S}_i{}^{lj} - \mathfrak{S}_i{}^{jl}\right) - \frac{1}{2}\partial_n\left(Y^{jl}{}_i{}^n + Y_i{}^{ljn} - Y_i{}^{jln}\right) ,$$

and the relocalized e.-m., $\mathfrak{t}_i{}^j := \widehat{\mathfrak{T}}_i{}^j$, with $\widehat{\mathfrak{S}}_{kl}{}^j = 0$, $Y_{ij}{}^{kl} = 0$, reads

$$\mathfrak{t}_i{}^j = \mathfrak{T}_i{}^j + \frac{1}{2}\partial_k\left(\mathfrak{S}^{jk}{}_i + \mathfrak{S}_i{}^{kj} - \mathfrak{S}_i{}^{jk}\right) .$$

- Belinfante tensor is *not symmetric*, in general,

$$2\mathfrak{t}_{[kl]} \equiv \frac{\delta\mathcal{L}}{\delta\Psi^A}(f_{kl})^A{}_B\Psi^B .$$

- The Gordon relocalization, mentioned above, *differs* from the Belinfante relocalization.

- It is crucial to distinguish closed and open systems. *Closed* system: dynamics totally determined by $\Psi^A(x)$. *Open* system: some fields are non-dynamical—*external*.
- Suppose all fields $\Psi^A(x)$ dynamical, that is, the system is closed. Then $\delta\mathcal{L}/\delta\Psi^A = 0$ and energy-momentum and angular momentum are *conserved*

$$\partial_j \mathfrak{T}_i{}^j = 0, \qquad \partial_j \mathfrak{J}_{kl}{}^j = 0\,.$$

Here $\mathfrak{J}_{kl}{}^j = \mathfrak{S}_{kl}{}^j + \mathfrak{L}_{kl}{}^j$ with $\mathfrak{L}_{kl}{}^j = x_k\,\mathfrak{T}_l{}^j - x_l\,\mathfrak{T}_k{}^j$.

- Canonical energy-momentum is asymmetric, in general,

$$2\mathfrak{T}_{[kl]} = \partial_j \mathfrak{S}_{kl}{}^j \neq 0\,.$$

The *Belinfante* tensor for a closed system is *conserved* and *symmetric*

$$\boxed{\partial_j \mathfrak{t}_i{}^j = 0, \qquad \mathfrak{t}_{[kl]} = 0\,.}$$

- Dynamics of open system (temporal change of state) is not determined by fundamental field variables, but also depends on “external” fields (background)
- Both canonical tensor $\mathfrak{T}_i{}^j$ and Belinfante tensor $\mathfrak{t}_i{}^j$ are *neither symmetric nor conserved, in general*

  We separate fields in two subsets $\Psi^A = \{\Psi^{\mathcal{A}}_{\rm dyn}, \Psi^{\alpha}_{\rm ext}\}$
- For external fields we have $\delta\mathcal{L}/\delta\Phi^A_{\rm ext} \neq 0$. The Noether identities reduce to *balance equations*

$$\partial_j \mathfrak{t}_i{}^j = -\frac{\delta\mathcal{L}}{\delta\Psi^{\alpha}_{\rm ext}}\partial_i\Psi^{\alpha}_{\rm ext}, \qquad 2\mathfrak{t}_{[kl]} = \frac{\delta\mathcal{L}}{\delta\Psi^{\alpha}_{\rm ext}}(f_{kl})^{\alpha}{}_{\beta}\Psi^{\beta}_{\rm ext}$$

- Balance relations yield *conservation laws* for background with symmetries. External fields are, e.g., constant in time/space

$$\frac{\partial\Psi^{\alpha}_{\rm ext}}{\partial t} = 0, \qquad \text{or/and} \qquad \partial_a\Psi^{\alpha}_{\rm ext} = 0$$

- The old dispute on the Abraham versus Minkowski energy-momentum tensor for matter in the electromagnetic field can be resolved by these methods, see Yuri Obukhov et al., loc. cit., and our joint book.

# 6. Dynamic Hilbert energy-momentum in general relativity

- How can we choose amongst the multitude of relocalized energy-momentum tensors, and how can we find the physical correct one? The Belinfante recipe was to kill $\mathfrak{T}_{[kl]}$. This does not yield a unique relocalized tensor.
- Hilbert defined already in 1915 the dynamic energy-momentum as the response of the matter Lagrangian to the variation of the metric:

$$^{\mathrm{Hi}}\mathfrak{t}_{ij} := 2\frac{\delta\mathfrak{L}(g,\Psi,\overset{\{\}}{\nabla}\Psi)}{\delta g^{ij}}\,.$$

  $g^{ij}$ (or its reciprocal $g_{kl}$) is the gravitational potential in Einstein's theory of gravitation (general relativity, GR). The matter Lagrangian is supposed to be *minimally coupled* to $g^{ij}$, in accordance with the equivalence principle. Only in the gravitational theory, in which spacetime can be deformed, we find a real local definition of the material energy-momentum tensor (see Weyl).
- The Hilbert definition is analogous to the relation from elasticity theory

$$\textit{stress} \sim \delta(\textit{elastic energy})/\delta(\textit{strain})\,.$$

  Recall that strain $\varepsilon^{ij} := \frac{1}{2}(^{(\mathrm{def})}g^{ij} - {}^{(\mathrm{undef})}g^{ij})$. Even the factor 2 is reflected in the Hilbert formula.

- Rosenfeld (1940) has shown, via Noether type theorems, that the Belinfante tensor $\mathfrak{t}_{ij}$, derived within special relativity, coincides with the Hilbert tensor ${}^{\rm Hi}\mathfrak{t}_{ij}$ of GR. Thus, the Belinfante-Rosenfeld recipe leads, *in the framework of GR,* to the energy-momentum tensor:

$$ {}^{\rm Hi}\mathfrak{t}_i{}^j = \mathfrak{t}_i{}^j = \mathfrak{T}_i{}^j + \frac{1}{2}\partial_k\left(\mathfrak{S}^{jk}{}_i + \mathfrak{S}_i{}^{kj} - \mathfrak{S}_i{}^{jk}\right) . \qquad (*) $$

  Recall that $(\mathfrak{T}_i{}^j, \mathfrak{S}_{ij}{}^k)$ are the canonical Noether currents.

- The Rosenfeld formula (*) identifies the Belinfante with the Hilbert tensor. In other words, the Belinfante tensor provides the correct source for Einstein's field equation.

- As long as we accept GR as the correct theory of gravity, the localization of energy-momentum and spin of matter is solved. This state of mind is conventionally kept till today by most theoretical physicists.

  One should note that the spin of matter has a rather auxiliary function in this approach. After all, the spin of the Hilbert-Belinfante-Rosenfeld tensor vanishes.

- However, the Sciama-Kibble theory of gravity (1961), generally known as Einstein-Cartan theory (EC), has turned the Rosenfeld formula (*) upside down...

# 7. Dynamic Sciama-Kibble spin in Poincaré gauge theory

- Gauging of the Poincaré group, gauge potentials orthonormal coframe and Lorentz connection ($\vartheta^\alpha = e_i{}^\alpha dx^i, \Gamma^{\alpha\beta} = \Gamma_i{}^{\alpha\beta} dx^i = -\Gamma^{\beta\alpha}$) $\Longrightarrow$ Poincaré gauge theory of gravity (PG) with a Riemann-Cartan space with Cartan's torsion and with Riemann-Cartan curvature, respectively:

$$C_{ij}{}^\alpha := D_{[i} e_{j]}{}^\alpha , \; R_{ij}{}^{\alpha\beta} := \text{“}D\text{”}_{[i} \Gamma_{j]}{}^{\alpha\beta} \qquad (or\ C^\alpha = D\vartheta^\alpha , \; R^{\alpha\beta} = \text{“}D\text{”}\Gamma^{\alpha\beta}).$$

- It is now straightforward: The currents are defined by variations with respect to the potentials:

$$^{\rm SK}\mathfrak{T}_\alpha{}^i = \frac{\delta \mathfrak{L}(e, \Gamma, \Psi, \overset{\Gamma}{D}\Psi)}{\delta e_i{}^\alpha}, \qquad {}^{\rm SK}\mathfrak{S}_{\alpha\beta}{}^i = \frac{\delta \mathfrak{L}(e, \Gamma, \Psi, \overset{\Gamma}{D}\Psi)}{\delta \Gamma_i{}^{\alpha\beta}}$$

  The dynamical definition of spin ${}^{\rm SK}\mathfrak{S}_{\alpha\beta}{}^i$ is due to Sciama-Kibble (1961). It is only possible in the Riemann-Cartan spacetime of PG. Also for energy-momentum ${}^{\rm SK}\mathfrak{T}_\alpha{}^i$ we have a new, revised definition. The Hilbert tensor plays no longer a decisive role.

- The Sciama-Kibble definition of the spin, is analogous to the relation

$$\textit{moment stress} \sim \delta(\textit{elastic energy})/\delta(\textit{contortion}) .$$

  Recall that the contortion is some kind of “rotational strain” in a Cosserat medium, see H & Obukhov, *Elie Cartan's torsion in geometry and in field theory, an essay*, `arXiv:0711.1535`.

- Application of Noether identities yields, after a lot of algebra,

$$^{\mathrm{SK}}\mathfrak{T}_\alpha{}^i = \mathfrak{T}_\alpha{}^i\,, \qquad ^{\mathrm{SK}}\mathfrak{S}_{\alpha\beta}{}^i = \mathfrak{S}_{\alpha\beta}{}^i\,.$$

  The dynamically defined currents à la Sciama-Kibble coincide with the canonical Noether currents of classical field theory, in marked contrast to the doctrine in the context of GR.

- We express the canonical energy-momentum tensor in the Hilbert one:

$$\begin{aligned} ^{\mathrm{SK}}\mathfrak{T}_\alpha{}^i &= {}^{\mathrm{Hi}}\mathfrak{t}_\alpha{}^i - \frac{1}{2}\overset{*}{D}_k\left(\mathfrak{S}_\alpha{}^{ik} - \mathfrak{S}^{ik}{}_\alpha + \mathfrak{S}^k{}_\alpha{}^i\right), \qquad (**) \\ ^{\mathrm{SK}}\mathfrak{S}_{\alpha\beta}{}^i &= \mathfrak{S}_{\alpha\beta}{}^i\,. \end{aligned}$$

  The new Rosenfeld type formula (**) reverses its original meaning in (*). In the Poincaré gauge theory (PG), the canonical tensor represents the energy-momentum distribution of matter and the (sym)metric Hilbert tensor now plays an auxiliary role. Moreover, we are now provided with a dynamic definition of the canonical spin tensor. In GR, the spin was only a kinematic quantity floating around freely.

- These results one the correct distribution of energy-momentum and spin in the framework of PG are are *independent* of a specific choice of the gravitational Lagrangian.

- However, if we choose the RC curvature scalar as a gravitational Lagrangian, we arrive at the Einstein-Cartan(-Sciama-Kibble) theory of gravitation, which is a viable theory of gravity competing with GR.

# 8. Extra dilation invariance and improved energy-momentum current

- Matter Lagrangian is assumed to be, in addition to Poincaré invariance, scale invariant, then we have the canon. Noether dilation current (3-form)

$$\Delta = \Delta^{\alpha}\eta_{\alpha}\,, \qquad \underbrace{\Delta}_{\text{intrinsic dil. curr.}} := w\Psi \wedge \frac{\partial L}{\partial D\Psi}$$

(here $\eta_{\alpha} := e_{\alpha}\rfloor\eta$, with frame $e_{\alpha}$ and volume 4-form $\eta$). The dilation current $\Delta^{\alpha}$ is somewhat analogous to the electric current $J^{\alpha}$ (1-parameter gauge transformation).

w is weight of scale transformation:

$$\Psi(x) \rightarrow \Psi'(x') = (e^{\omega})^{w}\Psi(e^{\omega}x)\,.$$

Noether law: $D\Delta + \vartheta^{\alpha} \wedge \mathfrak{T}_{\alpha} \stackrel{*}{=} D(\Delta + \underbrace{x^{\alpha} \wedge \mathfrak{T}_{\alpha}}_{\text{orb. dil. curr.}}) = 0\,.$

- Three types of Noether theorems (Poincaré $\otimes$ dilation):

$$\begin{aligned} D\mathfrak{T}_{\alpha} &= 0 && \text{(4 cons. momentum currents)}\,,\\ D\mathfrak{S}_{\alpha\beta} + \vartheta_{[\alpha} \wedge \mathfrak{T}_{\beta]} &= 0 && \text{(6 cons. angular momentum currents)}\,,\\ D\Delta \quad + \vartheta^{\alpha} \wedge \mathfrak{T}_{\alpha} &= 0 && \text{(1 cons. dilation current)} \end{aligned}$$

(in the literature, intrinsic and orbital dil. current are not cleanly defined).

- Take the superpotential 2-forms $M_\alpha, Y_{\alpha\beta}, Z$ such that

$$\begin{aligned}
\widehat{\mathfrak{T}}_\alpha(M) &= \mathfrak{T}_\alpha - DM_\alpha\,,\\
\widehat{\mathfrak{S}}_{\alpha\beta}(M,Y) &= \mathfrak{S}_{\alpha\beta} - \vartheta_{[\alpha}\wedge M_{\beta]} - DY_{\alpha\beta}\,,\\
\widehat{\Delta}(M,Z) &= \Delta - \vartheta^\alpha\wedge M_\alpha - DZ\,.
\end{aligned}$$

The hatted quantities fulfill again the $4+6+1$ conservation laws. The total charges remain the same.

- For the improved energy-momentum tensor $\not{t}_\alpha$ of Chernikov-Tagirov (1968) and Callan-Coleman-Jackiw (1970),

$$\not{t}_\alpha := \widehat{\mathfrak{T}}_\alpha(M) \qquad \text{for} \qquad \widehat{\mathfrak{S}}_{\alpha\beta}(M,Y) \stackrel{!}{=} 0 \quad \text{and} \quad \widehat{\Delta}(M,Z) \stackrel{!}{=} 0\,,$$

we require additionally that its trace vanishes ($\rightarrow$ soft pions):

$$\vartheta^\alpha\wedge \not{t}_\alpha = \vartheta^\alpha\wedge\widehat{\mathfrak{T}}_\alpha + D\Delta - DDZ \stackrel{!}{=} 0\,.$$

This can be achieved and, accordingly, the improved energy-momentum tensor is symmetric, traceless, and divergencefree:

$$\boxed{\vartheta_{[\alpha}\wedge \not{t}_{\beta]} = 0\,, \qquad \vartheta^\alpha\wedge \not{t}_\alpha = 0 \qquad D\not{t}_\alpha = 0\,.}$$

(In Ricci calculus: $\not{t}_{[\alpha\beta]} = 0\,, \quad \not{t}_\gamma{}^\gamma = 0\,, \quad \nabla_\beta \not{t}_\alpha{}^\beta = 0$.)

- $\mathfrak{T}_\alpha$ , $\mathfrak{S}_{\alpha\beta}$, and, for massless fields, additionally $\Delta$ are the inertial (and thus the gravitational) currents.

## 9. An algebra of the momentum and the spin currents?

- I discussed exclusively classical field theory. Can we learn something for a corresponding quantization of gravity? Our classical analysis has led us to the gravitational currents $\mathfrak{T}_\alpha$ and $\mathfrak{S}_{\alpha\beta}$. They represent the sources of gravity.
- In strong and in electroweak interaction, before the standard model had been worked out, one started with the current algebra of the phenomenologically known *strong* and the *electroweak* currents (Gell-Mann 1961, see also T.Y. Cao, *From Current Algebra to Quantum Chromodynamics,* Cambridge 2010).
- Schwinger (1963) studied, e.g., the equal time commutators of the components of the Hilbert e.-m. tensor. Should one try to include also the spin tensor components and turn to the canonical tensors?
- In the Sugawara model (1968), *A field theory of currents,* 8 vector and 8 axial vector currents for strong interaction are introduced and a symmetric e.-m. current expressed bilinearly in terms of these currents. Now that we have good arguments that the gravitational currents are $\mathfrak{T}_\alpha$ and $\mathfrak{S}_{\alpha\beta}$, one may want to develop a corresponding current algebra by determining the equal time commutator of these currents.....

Soli Deo Gloria