Vorlesung Mathematik, Blatt 4

(1) Stammfunktionen F zu f.

(a) $F(x) = \frac{a_1}{2}x^2 + \frac{a_2}{3}x^3 + \frac{a_3}{4}x^4 + \frac{a_4}{13}x^{13} + \text{const.}$

(b) $F(x) = \cosh x + \text{const.}$

(c) $F(x) = b \sin x + \text{const.}$

(d) $F(x) = a \tan x + \text{const.}$

(e) $F(x) = \frac{1}{2}\sin^2(ax) + \text{const.}$ [Substitution $u = \sin(ax)$]

(f) $F(x) = \frac{a}{b}\sinh(bx) + \frac{b}{a}\cosh(ax) + \text{const.}$

(g) $F(x) = \int dx\, [1-\sin(bx)]^n \cos(bx)$

Substitution: $u = 1-\sin(bx)$; $du = -dx\, b\cos(bx)$

$\Rightarrow F(x) = \int du \left(-\frac{1}{b}\right) u^n = -\frac{u^{n+1}}{(n+1)b} + \text{const}$

$= -\frac{1}{b(n+1)}[1-\sin(bx)]^{n+1} + \text{const.}$

(2) $I_n = \int_0^\infty dx\, x^n \exp(-ax)$

$u(x) = x^n$; $v'(x) = \exp(-ax)$

$u'(x) = nx^{n-1}$; $v(x) = -\frac{1}{a}\exp(-ax)$

$$I_n = \underbrace{-\frac{x^n}{a}\exp(-ax)\Big|_0^\infty}_{0} + \underbrace{\frac{n}{a}\int_0^\infty dx\, x^{n-1}\exp(-ax)}_{n/a\cdot I_{n-1}(x)}$$

$\Rightarrow \boxed{I_n = \frac{n}{a} I_{n-1}}$ für $n \geq 1$

$n=0$

$$I_0 = \int_0^\infty dx \exp(-ax) = -\frac{1}{a} \exp(-ax)\Big|_0^\infty = \frac{1}{a}$$

$$\Rightarrow I_1 = \frac{1}{a} I_0 = \frac{1}{a^2}$$

$$I_2 = \frac{2}{a} I_1 = \frac{2}{a^3}$$

Behauptung: $I_n = \dfrac{n!}{a^{n+1}}$

Beweis durch vollständige Induktion

$n=0 \Rightarrow I_0 = \dfrac{0!}{a^1} = \dfrac{1}{a}$ ✓

Falls Behauptung für $n=k \geq 0$ korrekt folgt

$$I_{k+1} = \frac{k+1}{a} I_k = \frac{k+1}{a} \cdot \frac{k!}{a^{k+1}} = \frac{(k+1)!}{a^{k+2}}$$ ✓

(b) $\int_a^b dx \cos^m(x)$

$u(x) = \cos^{m-1}(x)$; $u'(x) = -(m-1)\cos^{m-2}x \sin x$

$v'(x) = \cos x$; $v(x) = \sin x$

$$\Rightarrow \int_a^b dx \cos^m x = \cos^{m-1}x \sin x\Big|_a^b$$

$$+ (m-1) \int_a^b dx \cos^{m-2} x \underbrace{\sin^2 x}_{1-\cos^2 x}$$

$$\int_a^b dx \cos^m x = \cos^{m-1} x \sin x \Big|_a^b$$

$$+ (m-1) \int_a^b dx \left[\cos^{m-2} x - \cos^m x \right]$$

Bringe Term mit $\cos^m x$ auf linke Seite

$$\Rightarrow m \int_a^b dx \cos^m x = \cos^{m-1} x \sin x \Big|_a^b + (m-1) \int_a^b dx \cos^{m-2} x$$

(c) $$6 \int_0^\pi dx \cos^6 x = \underbrace{\cos^5 x \sin x \Big|_0^\pi}_{0} + 5 \int_0^\pi dx \cos^4 x$$

$$\Rightarrow \int_0^\pi dx \cos^6 x = \frac{5}{6} \int_a^b dx \cos^4 x$$

$$= \frac{5}{6} \cdot \frac{3}{4} \int_0^\pi dx \cos^2 x$$

$$= \frac{5 \cdot \cancel{3} \cdot 1}{2\cancel{6} \cdot 4 \cdot 2} \int_0^\pi dx = \frac{5!!}{6!!} \pi = \frac{5}{16} \pi$$