(1) $z_1^* = x_1 - iy_1$

$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$

$z_1 - z_2 = (x_1 - x_2) + i(y_1 - y_2)$

$z_1 \cdot z_2 = (x_1 + iy_1)(x_2 + iy_2) = x_1x_2 + ix_1y_2 + iy_1x_2 + i^2y_1y_2$

$= (x_1x_2 - y_1y_2) + i(x_1y_2 + y_1x_2)$

$$\frac{z_1}{z_2} = \frac{x_1 + iy_1}{x_2 + iy_2} = \frac{(x_1 + iy_1)(x_2 - iy_2)}{(x_2 + iy_2)(x_2 - iy_2)} = \frac{x_1x_2 - ix_1y_2 + ix_2y_1 + y_1y_2}{x_2^2 + y_2^2}$$

$$= \frac{x_1x_2 + y_1y_2 + i(x_2y_1 - x_1y_2)}{x_2^2 + y_2^2}$$

$$\Rightarrow \mathrm{Re}\left(\frac{z_1}{z_2}\right) = \frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2} \; ; \; \mathrm{Im}\left(\frac{z_1}{z_2}\right) = \frac{x_2y_1 - x_1y_2}{x_2^2 + y_2^2}$$

(2) $|z_1 z_2| = \sqrt{(x_1x_2 - y_1y_2)^2 + (x_1y_2 + y_1x_2)^2}$

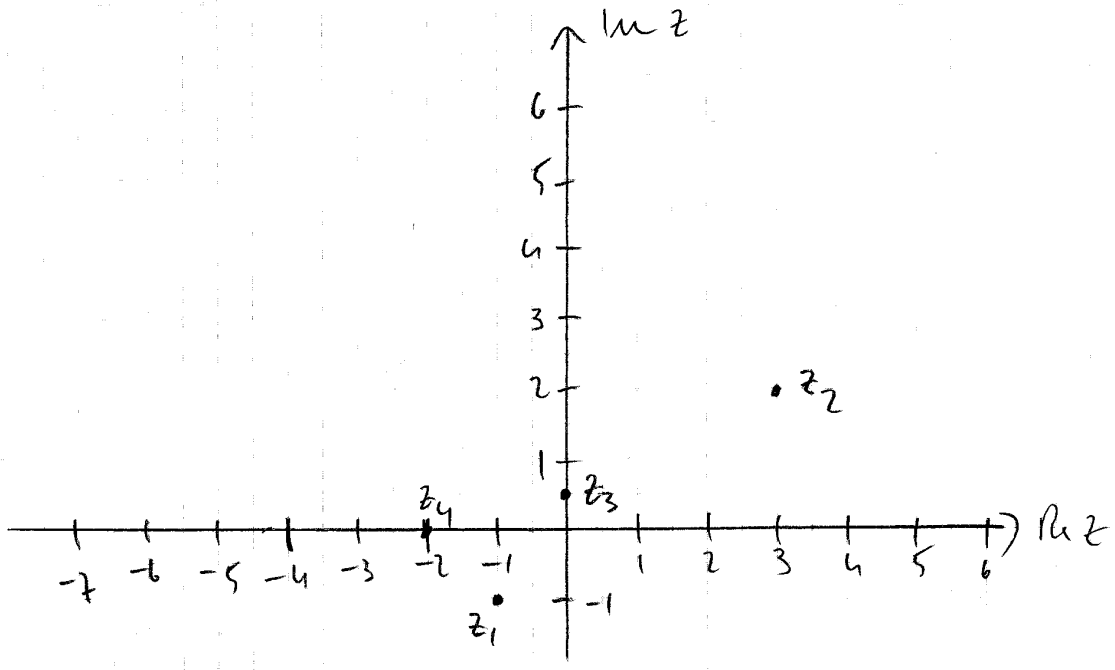
$= \sqrt{x_1^2x_2^2 - 2x_1x_2y_1y_2 + y_1^2y_2^2 + x_1^2y_2^2 + 2x_1y_1x_2y_2 + y_1^2x_2^2}$

$= \sqrt{x_1^2(x_2^2 + y_2^2) + y_1^2(x_2^2 + y_2^2)}$

$= \sqrt{(x_1^2 + y_1^2)(x_2^2 + y_2^2)} = \sqrt{x_1^2 + y_1^2} \cdot \sqrt{x_2^2 + y_2^2}$

$= |z_1| \cdot |z_2|$

(3)




(4) $(-i)^{4n} = \exp\left(-i \cdot \frac{\pi}{2} \cdot 4 \cdot n\right) = \exp(-2i\pi n) = 1$

für $n \in \mathbb{N}$

$(-i)^{4n+1} = (-i)^{4n}(-i) = -i$

$(-i)^{4n+2} = (-i)^{4n+1}(-i) = (-i)\cdot(-i) = -1$

$(-i)^{-17} = \frac{1}{(-i)^{17}} = \frac{1}{(-i)^{4\cdot 4+1}} = \frac{1}{-i} = i$

(5) (a)

$$z_1 = \frac{1+2i}{3+4i} = \frac{(1+2i)(3-4i)}{3^2+4^2} = \frac{3-4i+6i+8}{25}$$

$$= \frac{11+2i}{25}$$

$$z_2 = \frac{5+6i}{7+8i} = \frac{(5+6i)(7-8i)}{7^2+8^2} = \frac{35-40i+42i+48}{113}$$

$$= \frac{83+2i}{113}$$

$$z_1 + z_2 = \left(\frac{11}{25} + \frac{83}{113}\right) + 2i\left(\frac{1}{25} + \frac{1}{113}\right)$$

$$= \frac{1243+2075}{2825} + 2\frac{113+25}{2825}\,i$$

$$= \frac{3318+276i}{2825} \; ; \; |z_1+z_2| = \frac{\sqrt{11\,085\,300}}{2825} \simeq 1.1786$$

$$z_1 \cdot z_2 = \frac{(11+2i)(83+2i)}{25 \cdot 113} = \frac{11\cdot 83 - 4 + 2i(83+11)}{2825}$$

$$= \frac{909+188i}{2825} \; ; \; |z_1 \cdot z_2| = \frac{\sqrt{861625}}{2825} \simeq 0.329$$

$$\frac{z_1}{z_2} = \frac{113}{25} \cdot \frac{11+2i}{83+2i} = \frac{113}{25} \cdot \frac{(11+2i)(83-2i)}{83^2+4}$$

$$= \frac{113}{25} \cdot \frac{11\cdot 83 + 4 + 2i(83-11)}{6893} = \frac{\cancel{113}}{25} \cdot \frac{917+144i}{\cancel{6893}\;61}$$

$$= \frac{917+144i}{1525} \; ; \; \left|\frac{z_1}{z_2}\right| = \frac{\sqrt{861625}}{1525} \simeq 0.609$$

analog lassen sich auch (b) und (c) berechnen. Mit einem Computeralgebraprogramm Mathematica erhält man

(b) $z_1 = -\frac{18+13i}{17}$ ; $z_2 = \frac{14-6i}{29}$

$z_1 + z_2 = -\frac{284+479i}{493}$ ; $|z_1+z_2| = \frac{\sqrt{310097}}{493} \approx 1.130$

$z_1 \cdot z_2 = -\frac{330+74i}{493}$ ; $|z_1 \cdot z_2| = \frac{\sqrt{114376}}{493} \approx 0.686$

$\frac{z_1}{z_2} = -\frac{87+145i}{68}$ ; $\left|\frac{z_1}{z_2}\right| = \frac{\sqrt{28594}}{68} \approx 2.487$

(c) $z_1 = \frac{a+ib}{a-ib}$ ; $z_2 = \frac{b+ia}{b-ia} = \frac{i(a-ib)}{-i(a+ib)} = -\frac{1}{z_1}$

$z_1 = \frac{(a+ib)(a+ib)}{a^2+b^2} = \frac{a^2-b^2+2iab}{a^2+b^2}$

$z_2 = \frac{(b+ia)^2}{a^2+b^2} = \frac{b^2-a^2+2iab}{a^2+b^2}$

$z_1 + z_2 = \frac{4iab}{a^2+b^2}$ ; $z_1 \cdot z_2 = z_1 \cdot \left(-\frac{1}{z_1}\right) = -1$

$\frac{z_1}{z_2} = -z_1^2 = -\frac{(a^2-b^2+2iab)^2}{(a^2+b^2)^2} = -\frac{(a^2-b^2)^2 - 4a^2b^2 + 4iab(a^2-b^2)}{(a^2+b^2)^2}$

$= -\frac{a^4+b^4-6a^2b^2+4iab(a^2-b^2)}{(a^2+b^2)^2}$

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$|z_1+z_2| = \frac{4|ab|}{a^2+b^2}$ ; $|z_1 \cdot z_2| = 1$

$\left|\frac{z_1}{z_2}\right| = |z_1|^2 = \left|\frac{a+ib}{a-ib}\right|^2 = \frac{|a+ib|^2}{|a-ib|^2} = \frac{a^2+b^2}{a^2+(-b)^2} = \frac{a^2+b^2}{a^2+b^2} = 1$

denn $|z_1| = |z_2| = 1$ und $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$

(6) $z^2+z+1=0$

$\left(z+\frac{1}{2}\right)^2-\frac{1}{4}+1=0 \Rightarrow \left(z+\frac{1}{2}\right)^2=-\frac{3}{4}$

$\Rightarrow z+\frac{1}{2}=\pm\frac{\sqrt{3}}{2}i \Rightarrow z_{1/2}=\frac{-1\pm\sqrt{3}i}{2}$

$z^8=-1=\exp(i\pi+2\pi n i)\pm\exp[(2n+1)\pi i]\ ;\ n\in\mathbb{Z}$

$z_n=\exp\left[\frac{2n+1}{8}\pi i\right]\ ;\ n\in\{0,1,2,3,4,5,6,7\}$

(7) $z_1=1-2i\ ;\ |z_1|=\sqrt{1^2+2^2}=\sqrt{5}$

$\arg z_1=-\arccos\frac{1}{\sqrt{5}}+2\pi$

$z_2=3\ ;\ |z_2|=3\ ;\ \arg z_2=0$

$z_3=-17i\ ;\ |z_3|=17\ ;\ \arg z_3=\frac{3\pi}{2}$

$z_4=-4+3i\ ;\ |z_4|=\sqrt{4^2+3^2}=\sqrt{25}=5$

$\arg z_4=\arccos\left(-\frac{4}{5}\right)$

$z_5=1+\sqrt{3}i\ ;\ |z_5|=\sqrt{1^2+\sqrt{3}^2}=2\ ;\ \arg z_5=\arccos\frac{1}{2}=\frac{\pi}{3}$

(8) $z_1=30+40i\ ;\ |z_1|=10\cdot\sqrt{3^2+4^2}=50$

$\arg z_1=\arccos\left(\frac{3}{5}\right)=\varphi$

$\sqrt{z_1}=\pm\sqrt{50}\exp\left(\frac{i}{2}\varphi\right)=\pm5\sqrt{2}\exp\left(\frac{i}{2}\varphi\right)$

Es gilt

$\cos^2\frac{\varphi}{2}-\sin^2\frac{\varphi}{2}=\cos\varphi=\frac{3}{5}$

$\Rightarrow 2\cos^2\frac{\varphi}{2}=1+\frac{3}{5}=\frac{8}{5}\ ;\ \cos\frac{\varphi}{2}=\sqrt{\frac{4}{5}}=\frac{1}{5}\sqrt{20}=\frac{2}{5}\sqrt{5}$

$\sin\frac{\varphi}{2}=\sqrt{1-\frac{4}{5}}=\sqrt{\frac{1}{5}}=\frac{1}{5}\sqrt{5}$

$$\sqrt{z_1} = \pm 5\sqrt{2}\left(\frac{2}{5} + \frac{i}{5}\right)\sqrt{5}$$

$$z_1 = \pm (2+i)\sqrt{10}$$

$$z_2 = -9 + 12i \; ; \; |z_2| = \sqrt{81+144} = 15$$

$$\varphi = \arg z_2 = \arccos\left(-\frac{9}{15}\right) = \arccos\left(-\frac{3}{5}\right)$$

$$2\cos^2\frac{\varphi}{2} - 1 = -\frac{3}{5} \Rightarrow \cos^2\frac{\varphi}{2} = \frac{1}{5} \Rightarrow \cos\frac{\varphi}{2} = \frac{\sqrt{5}}{5}$$

$$\sin^2\frac{\varphi}{2} = \frac{4}{5} \; ; \; \sin\frac{\varphi}{2} = \frac{2}{5}\sqrt{5}$$

$$\Rightarrow \sqrt{z_2} = \pm\frac{\sqrt{15}\cdot\sqrt{5}}{5}(1+2i) = (1+2i)\sqrt{3}$$

(9) $A^* = \begin{pmatrix} -i & 1+i \\ 3-2i & -2i \end{pmatrix}$

$$A \cdot B = \begin{pmatrix} i & 1-i \\ 3+2i & 2i \end{pmatrix} \cdot \begin{pmatrix} 4+2i & -2i \\ 2 & 2+i \end{pmatrix}$$

$$= \left(\begin{array}{l|l} i(4+2i)+2(1-i) & i\cdot(-2i)+(1-i)(2+i) \\ (3+2i)(4+2i)+2\cdot 2i & (3+2i)(-2i)+2i(2+i) \end{array}\right)$$

$$= \left(\begin{array}{l|l} 4i-2+2-2i & 2+2+i-2i+1 \\ 12+6i+8i-4+4i & -6i+4+4i-2 \end{array}\right)$$

$$= \left(\begin{array}{l|l} 2i & 5-i \\ 8+18i & 2-2i \end{array}\right)$$

$$BA^* = \begin{pmatrix} 4+2i & -2i \\ 2 & 2+i \end{pmatrix}\begin{pmatrix} -i & 1+i \\ 3-2i & -2 \end{pmatrix}$$

$$= \left(\begin{array}{c|c} (4+2i)(-i) - 2i(3-2i) & (4+2i)(1+i) + (-2i)(-2) \\ -2i + (2+i)(3-2i) & 2(1+i) - 2i(2+i) \end{array}\right)$$

$$= \left(\begin{array}{c|c} -4i+2-6i-4 & 4+4i+2i-2-4 \\ -2i+6-4i+3i+2 & 2+2i-4i+2 \end{array}\right)$$

$$= \left(\begin{array}{c|c} -2-10i & -2+6i \\ 8-3i & 4-2i \end{array}\right)$$

$$B \cdot A = \left(\begin{array}{c|c} 4+2i & -2i \\ 2 & 2+i \end{array}\right)\begin{pmatrix} i & 1-i \\ 3+2i & 2i \end{pmatrix}$$

$$= \left(\begin{array}{c|c} (4+2i)i - 2i(3+2i) & (4+2i)(1-i) - 2i\cdot 2i \\ 2i + (2+i)(3+2i) & 2(1-i) + (2+i)2i \end{array}\right)$$

$$= \left(\begin{array}{c|c} 4i-2-6i+4 & 4-4i+2i+2+4 \\ 2i+6+4i+3i-2 & 2-2i+4i-2 \end{array}\right)$$

$$= \left(\begin{array}{c|c} 2-2i & 10-2i \\ 4+9i & 2i \end{array}\right)$$

$$[A,B] = AB - BA = \left(\begin{array}{c|c} 2i-(2-2i) & 5-i-(10-2i) \\ 8+18i-(4+9i) & 2-2i-2i \end{array}\right)$$

$$= \left(\begin{array}{c|c} -2+4i & -5+i \\ 4+9i & 2-4i \end{array}\right)$$