Relativistic Stokes friction

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November 21, 2024

1 The relativistic equation of motion

The non-relativistic equation of motion for a particle in a medium, taking into account the interaction with the medium in terms of a linear friction-force law ("Stokes friction") reads

$$\dot{\vec{p}} = m\dot{\vec{v}} = -\gamma(\vec{p} - m\vec{V}) = -\gamma m(\vec{v} - \vec{V}) = \gamma m(\vec{V} - \vec{v}), \tag{1}$$

where m is the mass of the particle, γ the friction coefficient, and $\vec{V}$ the velocity (field) of the medium. To find the relativistic generalization of this force law, we start with the covariant description in terms of the Minkowski force, K^μ:

$$\frac{\mathrm{d}p^\mu}{\mathrm{d}\tau} = K^\mu, \quad p^\mu = m\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} \tag{2}$$

with the proper time τ. This implies

$$p_\mu p^\mu = m^2c^2 = \text{const} \tag{3}$$

and thus

$$p_\mu \frac{\mathrm{d}p^\mu}{\mathrm{d}\tau} = p_\mu K^\mu = mcu_\mu K^\mu = 0. \tag{4}$$

For the following we define

$$u^\mu = \frac{1}{c}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} \Rightarrow p^\mu = mcu^\mu, \quad u_\mu u^\mu = 1. \tag{5}$$

The corresponding quantity of the fluid-flow field is

$$U^\mu = \frac{1}{\sqrt{1 - \vec{V}^2/c^2}}\begin{pmatrix} 1 \\ \vec{V}/c \end{pmatrix}. \tag{6}$$

The non-relativistic equation (1) must hold in the limit $|\vec{v}|, |\vec{V}| \ll c$. Together with the constraint (4) this leads to the ansatz for the friction Minkowski force:

$$K^\mu = mc\gamma[U^\mu - (u_\nu U^\nu)u^\mu]. \tag{7}$$

To check that this follows the usual conventions according to which the meaning of the "material constant" γ should be defined in the (local) rest frame of the medium, we write down the equation of motion in this frame, where $(U^\mu) = (1, 0, 0, 0)$. For the spatial components one obtains

$$\frac{\mathrm{d}\vec{p}}{\mathrm{d}\tau} = -mc\gamma u^0\vec{u} = -\gamma u^0\vec{p}. \tag{8}$$

The temporal component of the covariant equation of motion reads in this frame

$$\frac{\mathrm{d}p^0}{\mathrm{d}\tau} = mc\frac{\mathrm{d}u^0}{\mathrm{d}\tau} = mc\gamma[1-(u^0)^2] = -mc\gamma\vec{u}^2, \tag{9}$$

where in the final step we have used $(u^0)^2 = 1 + \vec{u}^2$. From this one finds

$$\frac{\mathrm{d}}{\mathrm{d}\tau}(u^0)^2 = 2u^0\frac{\mathrm{d}u^0}{\mathrm{d}\tau} = 2\vec{u}\cdot\mathrm{d}\vec{u}\mathrm{d}\tau \Rightarrow \frac{\mathrm{d}u^0}{\mathrm{d}\tau} = \frac{\vec{u}}{u^0}\cdot\frac{\mathrm{d}\vec{u}}{\mathrm{d}\tau}. \tag{10}$$

Thus multiplying (8) with $\vec{u}/u^0$ shows that (9) is fulfilled by the solution of (8), as it should be and is guaranteed a priori by the constraint (4).

Since $\mathrm{d}\tau = \mathrm{d}t/u^0$, the three-dimensional (non-covariant) version of the equation of motion in the rest frame of the medium reads

$$\frac{\mathrm{d}\vec{p}}{\mathrm{d}t} = -\gamma\vec{p}, \tag{11}$$

i.e., the friction coefficient as the usual meaning of the inverse relaxation time of the momentum of the particle in this frame, i.e.,

$$\vec{p}(t) = \vec{p}_0\exp(-\gamma t). \tag{12}$$

For the position of the particle we find

$$\frac{\mathrm{d}\vec{x}}{\mathrm{d}t} = \dot{\vec{x}} = \sqrt{1-\dot{\vec{x}}^2/c^2}\frac{\vec{p}_0}{m}\exp(-\gamma t). \tag{13}$$

Setting $\vec{p}_0 = p_0\vec{e}_1$ and $\vec{x}_0 = 0$ as the initial condition we find $x^2 = x^3 = 0 = \text{const}$ and setting $A = p_0/m$ from (13)

$$\dot{x}^1 = \sqrt{1-(\dot{x}^1/c)^2}A\exp(-\gamma t). \tag{14}$$

Taking the square of (14) yields

$$\dot{x}^1 = \frac{A\exp(-\gamma t)}{\sqrt{1+A^2/c^2\exp(-2\gamma t)}}. \tag{15}$$

Taking into account the initial condition $x^1(0) = 0$ integration results in

$$x^1(t) = \frac{c}{\gamma}\left[\operatorname{arsinh}\left(\frac{A}{c}\right) - \operatorname{arsinh}\left(\frac{A}{c}\exp(-\gamma t)\right)\right]. \tag{16}$$

The non-relativistic limit is found by assuming $|A/c| \ll 1$ and $\operatorname{arsinh} x = x + \mathcal{O}(x^3)$:

$$x^1(t) = \frac{A}{\gamma}[1-\exp(-\gamma t)], \tag{17}$$

which is the solution for the non-relativistic equation of motion

$$m\ddot{\vec{x}} = m\dot{\vec{v}} = -m\gamma\vec{v}. \tag{18}$$