Analytics of thermal Green's functions

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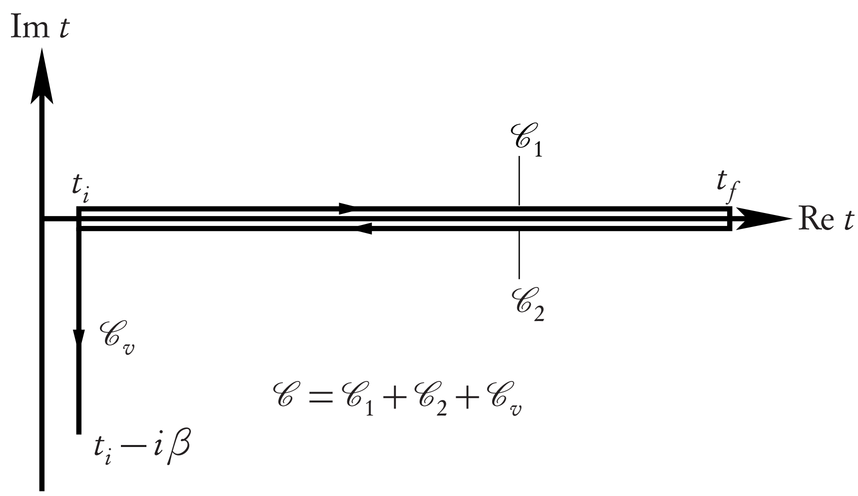
1 Conventions

We use the Minkowski metric, $\hat{\eta} = \mathrm{diag}(1, -1, -1, -1)$. Four-vectors are written as $\underline{x} = (x^0, \vec{x})$ and $\underline{x} \cdot \underline{y} = \eta_{\mu\nu} x^\mu y^\nu = x^0 y^0 - \vec{x} \cdot \vec{y}$ with $\vec{x}\vec{y} = x^1 y^1 + x^2 y^2 + x^3 y^3$. Integration measures are defined as $\mathrm{d}^4 p = \mathrm{d}p^0 \mathrm{d}p^1 \mathrm{d}p^2 \mathrm{d}p^3$ and $\mathrm{d}^3 p = \mathrm{d}p^1 \mathrm{d}p^2 \mathrm{d}p^3$. Further we use natural units with $\hbar = c = k_{\mathrm{B}} = 1$. Employing the spatio-temporal translation invariance of the equilibrium state, two-point correlation functions are written as $C(\underline{x}, \underline{y}) = \left\langle \mathbf{A}(\underline{x}) \mathbf{B}(\underline{y}) \right\rangle = \left\langle \mathbf{A}(\underline{x} - \underline{y}) \mathbf{B}(0) \right\rangle \equiv C(\underline{x} - \underline{y})$.

2 Green's functions for scalar fields

In the following we consider Green's functions on the extended Schwinger-Keldysh time-contour formalism for thermal equilibrium. We restrict ourselves to the most simple case of an uncharged scalar field, represented by Heisenberg-picture self-adjoint field operators, $\mathbf{\Phi}(t, \vec{x}) = \mathbf{\Phi}(\underline{x})$, for which $\mathbf{\Phi}^\dagger(\underline{x}) = \mathbf{\Phi}(\underline{x})$. We use the following definition of the extended Schwinger-Keldysh contour, which unifies the real- and imaginary-time (Matsubara) formalism in one framework.



In the following we assume $t_i \to -\infty$ and $t_f \to \infty$.
The contour Green's function is then defined by

$$\mathrm{i} G_{\mathcal{C}}(\underline{x}) = \langle \mathcal{T}_{\mathcal{C}} \mathbf{\Phi}(\underline{x}) \mathbf{\Phi}(0) \rangle \,, \tag{1}$$

where the field operators, which obey the usual bosonic equal-time commutator relations, are ordered according to the contour-time ordering symbol such that the field operator with the time argument "earlier" along the contour is to the right of the other field operator.

The expectation value is taken with respect to the equilibrium canonical statistical operator,

$$\mathbf{R} = \frac{1}{Z}\exp(-\beta\mathbf{H}), \quad Z = \operatorname{Tr}\exp(-\beta\mathbf{H}), \quad \langle\mathbf{A}\rangle = \operatorname{Tr}(\mathbf{RA}) = \operatorname{Tr}(\mathbf{AR}). \tag{2}$$

where $\mathbf{H}$ is the full Hamiltonian.

In the following we consider the relations between the different time ordering along the "chronological" branch, $\mathcal{C}_1$ and the "anti-chronological" branch $\mathcal{C}_2$ as well as the vertical branch $\mathcal{C}_v$:

$$\mathrm{i}G^{12}(t,\vec{x}) = \mathrm{i}G^{<}(t,\vec{x}) = \langle\mathbf{\Phi}(0)\mathbf{\Phi}(\underline{x})\rangle\,, \tag{3}$$

$$\mathrm{i}G^{21}(t,\vec{x}) = \mathrm{i}G^{>}(t,\vec{x}) = \langle\mathbf{\Phi}(\underline{x})\mathbf{\Phi}(0)\rangle\,, \tag{4}$$

$$\mathrm{i}G^{11}(t,\vec{x}) = \langle\mathcal{T}_c\mathbf{\Phi}(\underline{x})\mathbf{\Phi}(0)\rangle = \Theta(t)\mathrm{i}G^{>}(\underline{x}) + \Theta(-t)\mathrm{i}G^{<}(\underline{x}), \tag{5}$$

$$\mathrm{i}G^{22}(t,\vec{x}) = \langle\mathcal{T}_a\mathbf{\Phi}(\underline{x})\mathbf{\Phi}(0)\rangle = \Theta(t)\mathrm{i}G^{<}(\underline{x}) + \Theta(-t)\mathrm{i}G^{>}(\underline{x}), \tag{6}$$

$$\mathrm{i}G^{v}(-\mathrm{i}\tau,\vec{x}) = G_{\mathrm{M}}(\tau,\vec{x}) = \langle\mathbf{\Phi}(-\mathrm{i}\tau,\vec{x})\mathbf{\Phi}(0)\rangle = \mathrm{i}G^{>}(-\mathrm{i}\tau,\vec{x}). \tag{7}$$

The definition of G^v is meant for $\tau \in (0,\beta)$, and the Euclidean (Matsubara) propagator, G_{M}, is periodically continued to arbitrary $\tau\in\mathbb{R}$, $G_{\mathrm{M}}(\tau+\beta,\vec{x}) = G_{\mathrm{M}}(\tau,\vec{x})$.

The Heisenberg fields' time evolution reads

$$\mathbf{\Phi}(t,\vec{x}) = \exp(\mathrm{i}\mathbf{H}t)\mathbf{\Phi}(0,\vec{x})\exp(-\mathrm{i}\mathbf{H}t). \tag{8}$$

From this we derive

$$\begin{aligned}
\mathrm{i}G^{<}(t,\vec{x}) &= \frac{1}{Z}\operatorname{Tr}\left[\mathbf{\Phi}(0,\vec{0})\exp(-\beta\mathbf{H})\exp(\beta\mathbf{H})\mathbf{\Phi}(t,\vec{x})\exp(-\beta\mathbf{H})\right]\\
&= \frac{1}{Z}\operatorname{Tr}\left[\mathbf{\Phi}(0,\vec{0})\exp(-\beta\mathbf{H})\exp(\mathrm{i}\mathbf{H}(-\mathrm{i}\beta))\mathbf{\Phi}(t,\vec{x})\exp(-\mathrm{i}\mathbf{H}(-\mathrm{i}\beta))\right]\\
&= \frac{1}{Z}\operatorname{Tr}\left[\mathbf{\Phi}(0,\vec{0})\exp(-\beta\mathbf{H})\mathbf{\Phi}(t-\mathrm{i}\beta,\vec{x})\right]\\
&= \frac{1}{Z}\operatorname{Tr}\left[\exp(-\beta\mathbf{H})\mathbf{\Phi}(t-\mathrm{i}\beta,\vec{x})\mathbf{\Phi}(0,0)\right]\\
&= \mathrm{i}G^{>}(t-\mathrm{i}\beta,\vec{x}).
\end{aligned} \tag{9}$$

It is now convenient to express the Green's functions in terms of Fourier transforms. For the real-time branches, $i,j\in\{1,2\}$, we define

$$G^{ij}(t,\vec{x}) = \int_{\mathbb{R}^4}\frac{\mathrm{d}^4p}{(2\pi)^4}\exp(-\mathrm{i}\underline{x}\cdot\underline{p})\tilde{G}^{ij}(\underline{p}). \tag{10}$$

Then from (9) it is easy to show that the KMS condition (named after Kubo, Martin, and Schwinger)

$$\tilde{G}^{<}(\underline{p}) = \exp(-\beta p^0)\tilde{G}^{>}(\underline{p}) \tag{11}$$

holds.

Another important Green's function is the retarded Green's function,

$$\mathrm{i}G_{\mathrm{ret}}(\underline{x}) = \Theta(t)\,\langle[\mathbf{\Phi}(\underline{x}),\mathbf{\Phi}(0)]\rangle = \mathrm{i}\Theta(t)[G^{>}(\underline{x}) - G^{<}(\underline{x})] = G^{11}(\underline{x}) - G^{<}(\underline{x}). \tag{12}$$

Now in the sense of generalized functions

$$\Theta(t) = \int_{\mathbb{R}} \frac{\mathrm{d}p^0}{2\pi} \frac{\mathrm{i}}{p^0 + \mathrm{i}0^+} \exp(-\mathrm{i}p^0 t). \tag{13}$$

Using the convolution theorem from (12) it follows

$$\begin{aligned} \tilde{G}_{\mathrm{ret}}(p^0, \vec{p}) &= \int_{\mathbb{R}} \frac{\mathrm{d}p'^0}{2\pi} \frac{\mathrm{i}}{p^0 - p'^0 + \mathrm{i}0^+} \left[\tilde{G}^{>}(p'^0, \vec{p}) - \tilde{G}^{<}(p'^0, \vec{p})\right] \\ &= \int_{\mathbb{R}} \frac{\mathrm{d}p'^0}{2\pi} \frac{\rho(p'^0, \vec{p})}{p^0 - p'^0 + \mathrm{i}0^+}. \end{aligned} \tag{14}$$

From the definitions (3) and (4) it is easy to see that

$$[\mathrm{i}G^{<}(\underline{x})]^* = \mathrm{i}G^{>}(\underline{x}) = \mathrm{i}G^{<}(-\underline{x}), \quad [\mathrm{i}G^{>}(\underline{x})]^* = \mathrm{i}G^{<}(\underline{x}) = \mathrm{i}G^{>}(-\underline{x}) \tag{15}$$

For the Fourier transforms of these functions this implies

$$[\mathrm{i}\tilde{G}^{<}(\underline{p})]^* = \mathrm{i}\tilde{G}^{<}(\underline{p}), \quad [\mathrm{i}\tilde{G}^{>}(\underline{p})]^* = \mathrm{i}\tilde{G}^{>}(\underline{p}), \tag{16}$$

i.e., $\mathrm{i}\tilde{G}^{<}(\underline{p}) \in \mathbb{R}$ and $\mathrm{i}\tilde{G}^{>}(\underline{p}) \in \mathbb{R}$. Thus also the **spectral function** $\rho(\underline{p}) \in \mathbb{R}$.
From the relation

$$\frac{1}{p^0 - p'^0 + \mathrm{i}0^+} = \mathcal{P}\frac{1}{p^0 - p'^0} - \mathrm{i}\pi\delta(p^0 - p'^0), \tag{17}$$

where $\mathcal{P}$ denotes the Cauchy-principal value when used in the sense of a distribution in integrals over test functions, this leads to

$$\rho(\underline{p}) = -2\,\mathrm{Im}\,\tilde{G}_{\mathrm{ret}}(\underline{p}) \tag{18}$$

and the Kramers-Kronig relation

$$\mathrm{Re}\,G_{\mathrm{ret}}(\underline{p}) = -\mathcal{P}\int_{\mathbb{R}} \frac{\mathrm{d}p^0}{\pi} \frac{\mathrm{Im}\,G_{\mathrm{ret}}(p'^0, \vec{p})}{p^0 - p'^0}. \tag{19}$$

With the KMS relation (11) we find

$$\rho(\underline{p}) = \mathrm{i}\left[\tilde{G}^{>}(\underline{p}) - \tilde{G}^{<}(\underline{p})\right] = \mathrm{i}\tilde{G}^{>}(\underline{p})\left[1 - \exp(-\beta p^0)\right] \tag{20}$$

or

$$\mathrm{i}\tilde{G}^{>}(\underline{p}) = \rho(\underline{p})\frac{\exp(\beta p^0)}{\exp(\beta p^0) - 1} = \rho(\underline{p})\left[1 + f_{\mathrm{B}}(p^0)\right] \tag{21}$$

with the Bose-Einstein distribution

$$f_{\mathrm{B}}(p^0) = \frac{1}{\exp(\beta p^0) - 1}. \tag{22}$$

In a similar way one finds from (11)

$$\mathrm{i}\tilde{G}^{<}(\underline{p}) = \rho(\underline{p}) f_{\mathrm{B}}(p^0), \tag{23}$$

i.e., all real-time Green's functions are solely determined by the spectral function, $\rho(\underline{p})$, (21), (23), and (14).

Fourier-transforming (16) leads to

$$\left[\tilde{G}^{>}(\underline{p})\right]^{*} = -\tilde{G}^{<}(-\underline{p}). \tag{24}$$

Since $\rho(\underline{p}) \in \mathbb{R}$ we have

$$\rho(\underline{p}) = \rho^{*}(\underline{p}) = -\mathrm{i}\left[\tilde{G}^{>*}(\underline{p}) - \tilde{G}^{<*}(\underline{p})\right] = +\mathrm{i}\left[\tilde{G}^{<}(-\underline{p}) - \tilde{G}^{>}(-\underline{p})\right] = -\rho(-\underline{p}). \tag{25}$$

Further the equilibrium state (2) is invariant under rotations, and this is thus also true for the Green's functions, the $\tilde{G}^{ij}(\underline{p})$ depend only on $|\vec{p}|$, which with (25) implies that

$$\rho(-p^0, \vec{p}) = -\rho(p^0, \vec{p}). \tag{26}$$

Now we can also express the Matsubara Green's function in terms of the spectral function. To this end we use (7). Since by definition it is periodic in τ with period β we write it in terms of a corresponding Fourier series,

$$G_{\mathrm{M}}(\tau, \vec{x}) = \frac{1}{\beta} \sum_{n \in \mathbb{Z}} \int_{\mathbb{R}^3} \frac{\mathrm{d}^3 p}{(2\pi)^3} \exp(-\mathrm{i}\omega_n \tau) \exp(\mathrm{i}\vec{p} \cdot \vec{x}) \tilde{G}_{\mathrm{M}}(\mathrm{i}\omega_n, \vec{p}), \quad \omega_n = \frac{2\pi n}{\beta}, \quad n \in \mathbb{Z}. \tag{27}$$

On the other hand $G_{\mathrm{M}}(\tau, \vec{x}) = \mathrm{i}G^{>}(-\mathrm{i}\tau, \vec{x})$ and thus

$$G_{\mathrm{M}}(\tau, \vec{x}) = \mathrm{i}G^{>}(-\mathrm{i}\tau, \vec{x}) = \int_{\mathbb{R}^4} \frac{\mathrm{d}^4 p}{(2\pi)^4} \exp(-p^0 \tau) \exp(\mathrm{i}\vec{p} \cdot \vec{x}) \mathrm{i}\tilde{G}^{>}(p^0, \vec{p}). \tag{28}$$

With (21) we find

$$G_{\mathrm{M}}(\tau, \vec{x}) = \int_{\mathbb{R}^4} \frac{\mathrm{d}^4 p}{(2\pi)^4} \exp(-p^0 \tau) \exp(\mathrm{i}\vec{p} \cdot \vec{x}) \left[1 + f_{\mathrm{B}}(p^0)\right] \rho(\underline{p}). \tag{29}$$

To get the Fourier coefficients for the Fourier series wrt. τ we calculate, using (28) and (29)

$$\begin{aligned}
g_{\mathrm{M}}(\mathrm{i}\omega_n, \vec{x}) &= \int_0^{\beta} G_{\mathrm{M}}(\tau, \underline{x}) \exp(\mathrm{i}\omega_n \tau) \\
&= \int_{\mathbb{R}^4} \frac{\mathrm{d}^4 p}{(2\pi)^4} \exp(\mathrm{i}\vec{p} \cdot \vec{x}) \int_0^{\beta} \mathrm{d}\tau \exp[(-p^0 + \mathrm{i}\omega_n)\tau] \left[1 + f_{\mathrm{B}}(p^0)\right] \rho(\underline{p}) \\
&= \int_{\mathbb{R}^4} \frac{\mathrm{d}^4 p}{(2\pi)^4} \exp(\mathrm{i}\vec{p} \cdot \vec{x}) \frac{\rho(p^0, \vec{p})}{p^0 - \mathrm{i}\omega_n} \quad \Rightarrow \\
\tilde{G}_{\mathrm{M}}(\mathrm{i}\omega_n, \vec{p}) &= \int_{\mathbb{R}} \frac{\mathrm{d}p^0}{2\pi} \frac{\rho(p^0, \vec{p})}{p^0 - \mathrm{i}\omega_n} = \tilde{G}_{\mathrm{ret}}(\mathrm{i}\omega_n, \vec{p}).
\end{aligned} \tag{30}$$

Using this in the Fourier series gives

$$G_{\mathrm{M}}(\tau, \vec{x}) = \int_{\mathbb{R}^4} \frac{\mathrm{d}^4 p}{(2\pi)^4} \exp(\mathrm{i}\vec{p} \cdot \vec{x}) \rho(p^0, \vec{p}) F(p^0, \tau) \tag{31}$$

with

$$F(p^0, \tau) = \frac{1}{\beta} \sum_{n \in \mathbb{Z}} \frac{\exp(-\mathrm{i}\omega_n \tau)}{p^0 - \mathrm{i}\omega_n} = f_{\mathrm{B}}(p^0) \exp[p^0(\beta - \tau)], \quad 0 < \tau < \beta. \tag{32}$$

The latter result we prove in Appendix A. Using this in (31) leads to

$$\begin{aligned}
G_{\mathrm{M}}(\tau,\vec{x}) &= \int_{\mathbb{R}^4} \frac{\mathrm{d}^4 p}{(2\pi)^4} \exp(\mathrm{i}\vec{p}\cdot\vec{x})\rho(\underline{p}) f_{\mathrm{B}}(p^0)\exp[p^0(\beta-\tau)] \\
&= \int_{\mathbb{R}^3} \frac{\mathrm{d}^3 p}{(2\pi)^3} \left[\int_0^\infty \frac{\mathrm{d}p^0}{(2\pi)} \rho(p^0,\vec{p}) f_{\mathrm{B}}(p^0)\exp[p^0(\beta-\tau)] \right. \\
&\qquad \left. + \int_{-\infty}^0 \rho(p^0,\vec{p}) f_{\mathrm{B}}(p^0)\exp[p^0(\beta-\tau)] \right] \\
&= \int_{\mathbb{R}^3} \frac{\mathrm{d}^3 p}{(2\pi)^3} \int_0^\infty \frac{\mathrm{d}p^0}{2\pi} \left[\rho(p^0,\vec{p}) f_{\mathrm{B}}(p^0)\exp[p^0(\beta-\tau)] \right. \\
&\qquad \left. + \rho(-p^0,\vec{p}) f_{\mathrm{B}}(-p^0)\exp[-p^0(\beta-\tau)] \right] \\
&= \int_{\mathbb{R}^3} \frac{\mathrm{d}^3 p}{(2\pi)^3} \exp(\mathrm{i}\vec{p}\cdot\vec{x}) \int_0^\infty \frac{\mathrm{d}p^0}{2\pi} \rho(p^0,\vec{p}) \left[f_{\mathrm{B}}(p^0)\exp(\beta p^0)\exp(-p^0\tau) \right. \\
&\qquad \left. - f_{\mathrm{B}}(-p^0)\exp(-\beta p^0)\exp(p^0\tau) \right].
\end{aligned} \tag{33}$$

Further we have

$$f_{\mathrm{B}}(p^0)\exp(\beta p^0) = \frac{\exp(\beta p^0/2)}{2\sinh(\beta p^0)/2}, \quad f_{\mathrm{B}}(-p^0)\exp(-\beta p^0) = -\frac{\exp(-\beta p^0/2)}{2\sinh(\beta p^0/2)}. \tag{34}$$

Using this in (33) finally yields

$$G_{\mathrm{M}}(\tau,\vec{x}) = \int_{\mathbb{R}^3} \frac{\mathrm{d}^3 p}{(2\pi)^3} \int_0^\infty \frac{\mathrm{d}p^0}{2\pi} \exp(\mathrm{i}\vec{p}\cdot\vec{x})\rho(p^0,\vec{p}) \frac{\cosh(p^0\tau - \beta p^0/2)}{\sinh(\beta p^0/2)}. \tag{35}$$

3 Free scalar field

The above formalism can easily be used to find the various Green's functions for the free scalar field. The Lagrangian reads

$$\mathscr{L} = \frac{1}{2}(\partial_\mu \Phi)(\partial^\mu \Phi) - \frac{m^2}{2}\phi^2. \tag{36}$$

The equation of motion follows from the Euler-Lagrange equations

$$\partial_\mu \frac{\partial \mathscr{L}}{\partial(\partial_\mu \Phi)} = \Box \Phi = \frac{\partial \mathscr{L}}{\partial \Phi} = -m^2\Phi \;\Rightarrow\; (\Box + m^2)\Phi = 0, \tag{37}$$

i.e., the Klein-Gordon equation for free pqrticles.
The canonical field momentum is

$$\Pi = \frac{\partial \mathscr{L}}{\partial \dot{\Phi}} = \dot{\Phi}. \tag{38}$$

Now we use the canonical equal-time commutation relation for Bosonic fields,

$$[\mathbf{\Phi}(t,\vec{x}),\mathbf{\Phi}(t,\vec{y})] = 0, \quad [\mathbf{\Phi}(t,\vec{x}),\mathbf{\Pi}(t,\vec{y})] = \left[\mathbf{\Phi}(t,\vec{x}),\dot{\mathbf{\Phi}}(t,\vec{y})\right] = \mathrm{i}\delta^{(3)}(\vec{x}-\vec{y}). \tag{39}$$

Now we solve the equations (37) and (39) with the Fourier ansatz

$$\mathbf{\Phi}(t,\vec{x}) = \int_{\mathbb{R}^3} \frac{\mathrm{d}^3 p}{(2\pi)^3} \boldsymbol{\phi}(t,\vec{p}) \exp(\mathrm{i}\vec{x}\cdot\vec{p}). \tag{40}$$

The Klein-Gordon equation (37) leads to

$$\partial_t^2 \boldsymbol{\phi}(t,\vec{p}) = -E_p^2 \boldsymbol{\phi}(t,\vec{p}), \quad E_p = \sqrt{\vec{p}^2 + m^2}. \tag{41}$$

Together with the self-adjointness of $\mathbf{\Phi}(\underline{x})$ this leads to the general solution

$$\mathbf{\Phi}(t) = \int_{\mathbb{R}^3} \frac{\mathrm{d}^3 p}{(2\pi)^3} N(\vec{p}) \left[\mathbf{a}(\vec{p}) \exp(-\mathrm{i}\underline{x}\cdot\underline{p}) + \mathbf{a}^\dagger(\vec{p}) \exp(+\mathrm{i}\underline{x}\cdot\underline{p})\right]_{p^0=E_p}. \tag{42}$$

Since the equations (41) are that of independent harmonic oscillators, it is plausible to try the corresponding ansatz for the operators $\mathbf{a}$ and $\mathbf{a}^\dagger$ as annihilation and creation operators for indpendent-oscillator modes,

$$[\mathbf{a}(\vec{p}), \mathbf{a}(\vec{q})] = 0, \quad \left[\mathbf{a}(\vec{p}), \mathbf{a}^\dagger(\vec{q})\right] = \delta^{(3)}(\vec{p}-\vec{q}). \tag{43}$$

Using (42) and (43) in (39) leads to $N(\vec{p}) = 1/\sqrt{(2\pi)^3 2E_p}$, i.e., the mode decomposition for the field (42) finally reads

$$\mathbf{\Phi}(t) = \int_{\mathbb{R}^3} \frac{\mathrm{d}^3 p}{\sqrt{(2\pi)^3 2E_p}} \left[\mathbf{a}(\vec{p}) \exp(-\mathrm{i}\underline{x}\cdot\underline{p}) + \mathbf{a}^\dagger(\vec{p}) \exp(+\mathrm{i}\underline{x}\cdot\underline{p})\right]_{p^0=E_p}. \tag{44}$$

To get the spectral function we use (20). For this we need the commutator function of the free fields

$$\mathrm{i}\Delta(\underline{x}) = \langle[\mathbf{\Phi}(\underline{x}), \mathbf{\Phi}(0)]\rangle. \tag{45}$$

Using (44) and the commutator relations (43) we find after some algebra

$$\mathrm{i}\Delta(x) = \int_{\mathbb{R}^3} \frac{\mathrm{d}^3 p}{(2\pi)^3 2E_p} [\exp(-\mathrm{i}E_p t) - \exp(\mathrm{i}E_p t)] \exp(\mathrm{i}\vec{p}\cdot\vec{x}). \tag{46}$$

The four-dimensional Fourier transform leads to the spectral function

$$\rho(\underline{p}) = \int_{\mathbb{R}^4} \mathrm{d}^4 x \exp(\mathrm{i}\underline{x}\cdot\underline{p}) \mathrm{i}\Delta(\underline{x}) = \frac{2\pi}{2E_p}\left[\delta(p^0 - E_p) - \delta(p^0 + E_p)\right] = 2\pi\sigma(p^0)\delta(\underline{p}^2 - m^2). \tag{47}$$

With (14) we immediately get the retarded Green's function,

$$\tilde{\Delta}_{\text{ret}}(\underline{p}) = \frac{1}{p^2 - m^2 + \mathrm{i}\sigma(p^0)0^+}. \tag{48}$$

From (21) and (23) we find for the "fixed-order" real-time Green's functions

$$\mathrm{i}\tilde{\Delta}^<(\underline{p}) = 2\pi\sigma(p^0)\delta(p^2 - m^2) f_{\mathrm{B}}(p^0), \quad \mathrm{i}\tilde{\Delta}^>(\underline{p}) = 2\pi\sigma(p^0)\delta(p^2 - m^2)\left[1 + f_{\mathrm{B}}(p^0)\right]. \tag{49}$$

For the following it is convenient to use

$$\Theta(-p^0) f_{\mathrm{B}}(p^0) = \Theta(-p^0) f_{\mathrm{B}}(-|p^0|) = -\Theta(-p^0)\left[1 + f_{\mathrm{B}}(|p^0|)\right] \tag{50}$$

and thus

$$\sigma(p^0) f_{\mathrm{B}}(p^0) = \left[\Theta(p^0) - \Theta(-p^0)\right] f_{\mathrm{B}}(p^0) = \Theta(-p^0) + f_{\mathrm{B}}(|p^0|). \tag{51}$$

From this we can write (49) in the form

$$\mathrm{i}\tilde{\Delta}^{<}(\underline{p}) = 2\pi \left[\Theta(-p^0) + f_{\mathrm{B}}(|p^0|)\right] \delta(p^2 - m^2), \quad \mathrm{i}\tilde{\Delta}^{>}(\underline{p}) = 2\pi \left[\Theta(p^0) + f_{\mathrm{B}}(|p^0|)\right] \delta(p^2 - m^2). \tag{52}$$

Using (12) we find for the chronologically-ordered Green function

$$\begin{aligned}
\tilde{\Delta}^{11}(\underline{p}) &= \tilde{\Delta}_{\mathrm{ret}} + \tilde{\Delta}^{<}(\underline{p}) \\
&= \frac{1}{p^2 - m^2 + \mathrm{i}\sigma(p^0)0^+} - 2\pi\mathrm{i}\sigma(p^0)\delta(p^2 - m^2) f_{\mathrm{B}}(p^0) \\
&= \mathcal{P}\frac{1}{p^2 - m^2} - \mathrm{i}\pi\delta(p^2 - m^2)\left[\sigma(p^0) + 2\Theta(-p^0) + 2f_{\mathrm{B}}(|p^0|)\right] \\
&= \mathcal{P}\frac{1}{p^2 - m^2} - \mathrm{i}\pi\delta(p^2 - m^2) - 2\pi\mathrm{i}\delta(p^2 - m^2) f_{\mathrm{B}}(|p^0|) \\
&= \frac{1}{p^2 - m^2 + \mathrm{i}0^+} - 2\pi\mathrm{i}\delta(p^2 - m^2) f_{\mathrm{B}}(|p^0|)
\end{aligned} \tag{53}$$

and then with (52) and (53)

$$\begin{aligned}
\tilde{\Delta}^{22}(\underline{p}) &= \tilde{\Delta}^{>}(\underline{p}) + \tilde{\Delta}^{<}(\underline{p}) - \tilde{\Delta}^{11}(\underline{p}) \\
&= -\frac{1}{p^2 - m^2 - \mathrm{i}0^+} - 2\pi\mathrm{i}\delta(p^2 - m^2) f_{\mathrm{B}}(|p^0|).
\end{aligned} \tag{54}$$

4 Thermodynamics of ideal gases

As a simple application of the Green's function techniques elaborated above, we shall calculate the thermodynamics of ideal gases. The natural approach is to evaluate the partition sum,

$$Z = \mathrm{Tr}\exp(-\beta\mathbf{H}), \quad \Omega = \ln Z. \tag{55}$$

The canonical statistical operator is then written as

$$\mathbf{R} = \exp(-\beta\mathbf{H} - \Omega), \tag{56}$$

and the entropy is then given by

$$S = -\langle \ln \mathbf{R} \rangle = \langle \beta\mathbf{H} \rangle + \Omega = \beta U + \Omega, \tag{57}$$

where U is the internal energy. In this formulation the indpendent variables are β and the "quantization volume" V, when more realistically assuming that we consider a gas in a finite volume. One approach is to use a cube of length L and impose periodic boundary conditions on the fields and only then take the thermodynamic limit $V \to \infty$ while keeping the densities of extensive quantities fixed. Here we use our analytic Green's functions to directly evaluate the energy density $u = U/V$ in the hithereto used direct calculation in the thermodynamic limit.

First, however, we briefly derive the connection between the statistical approach to thermodynamics, by using (57), using β and V as the independent variables. We find

$$\mathrm{d}S = \mathrm{d}\Omega + \beta\mathrm{d}U + U\mathrm{d}\beta = \mathrm{d}V\frac{\partial\Omega}{\partial V} + \mathrm{d}\beta\frac{\partial\Omega}{\partial\beta} + \beta\mathrm{d}U + U\mathrm{d}\beta. \tag{58}$$

Now from (55) we find

$$\frac{\partial \Omega}{\partial \beta} = \frac{1}{Z}\frac{\partial Z}{\partial \beta} = -\langle \mathbf{H} \rangle = -U \tag{59}$$

and thus

$$\mathrm{d}S = \mathrm{d}V\frac{\partial \Omega}{\partial V} + \beta \mathrm{d}U \ \Rightarrow \ \mathrm{d}U = T\mathrm{d}S - \frac{\partial (T\Omega)}{\partial V}\mathrm{d}V \quad \text{with} \quad T = \frac{1}{\beta}. \tag{60}$$

From this we find, using the thermodynamical relation

$$\mathrm{d}U = T\mathrm{d}S - p\mathrm{d}V \ \Rightarrow \ p = \frac{\partial (T\Omega)}{\partial V} = \frac{T\Omega}{V}. \tag{61}$$

This shows that Ω is the Landau thermodynamic potential, cf. (57) related to the entropy by the Legendre transformation

$$\Omega = \frac{pV}{T} = S - \frac{U}{T}. \tag{62}$$

For the free Bose field treated in the previous section we can evaluate Ω by first evaluating the energy density, u. Using the Lagrangian (36) we find for the Hamilton density

$$\mathcal{H} = \Pi\dot{\Phi} - \mathscr{L} = \frac{1}{2}(\dot{\Phi}^2 + (\vec{\nabla}\Phi)^2 + m^2\Phi^2). \tag{63}$$

The internal-energy density is given by the thermal expectation value of $\mathcal{H}$. Since for the here considered case of the free field, i.e., the ideal gas of uncharged bosons, the Hamilton density is quadratic in the field and its derivatives, we can express it using the free Green's function

$$\mathrm{i}\Delta^{>}(x-y) = \left\langle \mathbf{\Phi}(\underline{x})\mathbf{\Phi}(\underline{y})\right\rangle \ \Rightarrow \ u = \langle \boldsymbol{\mathcal{H}} \rangle = \frac{1}{2}(\partial_x^0\partial_y^0 + \vec{\nabla}_x \cdot \vec{\nabla}_y + m^2)\mathrm{i}\Delta^{>}(x-y)\Big|_{\underline{y}=\underline{x}}. \tag{64}$$

This we can rewrite as

$$u = \frac{1}{2}(-\partial_x^0 - \vec{\nabla}_x^2 + m^2)\mathrm{i}\Delta^{>}(x)\Big|_{\underline{x}=0}. \tag{65}$$

Using the Fourier representation this translates into

$$u = \frac{1}{2}\int_{\mathbb{R}^4} \frac{\mathrm{d}^4 p}{(2\pi)^4}\mathrm{i}\tilde{\Delta}^{>}(p)[(p^0)^2 + \vec{p}^2 + m^2]. \tag{66}$$

Now using (52) and doing the p^0 integral leads to

$$u = \int_{\mathbb{R}^3} \frac{\mathrm{d}^3 p}{(2\pi)^3} E_p \left[\frac{1}{2} + f_{\mathrm{B}}(E_p)\right]. \tag{67}$$

Obviously the contribution from the factor 1/2 in the square bracket is divergent. We see that this divergent contribution originates from the vacuum energy, $u_0 = u|_{T\to 0^+} = u|_{\beta\to\infty}$. As usual in relativistic quantum-field theory we have to renormalize the Hamiltonian such that the vacuum energy becomes finite, and the usual choice is to make the vacuum energy zero. Thus we find

$$u_{\mathrm{ren}} = u - u_0 = \int_{\mathbb{R}^3} \frac{\mathrm{d}^3 p}{(2\pi)^3} E_p f_{\mathrm{B}}(E_p). \tag{68}$$

As expected from the translation invariance of the equilibrium state in the thermodynamic limit, this energy density does not depend on $\underline{x}$, and we have to consider it as the density of a gas contained in a very large volume, V. The total energy thus is

$$U = V u_{\mathrm{ren}} = V \int_{\mathbb{R}^3} \frac{\mathrm{d}^3 p}{(2\pi)^3} E_p f_{\mathrm{B}}(E_p) = -\frac{\partial \Omega}{\partial \beta}. \tag{69}$$

We can use this to evaluate $\Omega = \ln Z$, using that $\Omega \to 0$ for $T \to 0^+$, assuming that we have renormalized the vacuum energy to 0, such that $\mathbf{H}$ is a positive definite operator. This leads to

$$\Omega = \ln Z = -V \int_{\mathbb{R}^3} \frac{\mathrm{d}^3 p}{(2\pi)^3} \ln[1 - \exp(-\beta E_p)], \tag{70}$$

where we used

$$\partial_\beta \ln[1 - \exp(-\beta E_p)] = -E_p \frac{\exp(-\beta E_p)}{1 - \exp(-\beta E_p)} = -E_p f_{\mathrm{B}}(E_p). \tag{71}$$

A Derivation of (32)

To prove (32) we note that

$$(\partial_\tau + p^0) F(p^0, \tau) = \frac{1}{\beta} \sum_{n \in Z} \exp(-\mathrm{i} \omega_n \tau) = \sum_{n \in \mathbb{Z}} \delta(\tau - n\beta). \tag{72}$$

Now we consider $F(p^0, \tau)$ for $\tau \in (-\beta, \beta)$. In this interval for τ we have

$$(\partial_\tau + p^0) F(p^0, \tau) = \delta(\tau), \quad \tau \in (-\beta, \beta). \tag{73}$$

This equation is obviously solved by

$$F(p^0, \tau) = [A_< \Theta(-\tau) + A_> \Theta(\tau)] \exp(-p^0 \tau), \quad \tau \in (-\beta, \beta). \tag{74}$$

Integrating (73) over an infinitesimal interval around 0 yields

$$F(p^0, 0^+) - F(p^0, 0^-) = A_> - A_< = 1. \tag{75}$$

Further from $F(p^0, \tau - \beta) = F(p^0, \tau)$ for $\tau \in (0, \beta)$ we find

$$A_< \exp(\beta p^0) = A_> \stackrel{(75)}{\Rightarrow} A_> = \exp(\beta p^0) f_{\mathrm{B}}(p^0), \tag{76}$$

and thus with (74)

$$F(p^0, \tau) = f_{\mathrm{B}}(p^0) \exp[p^0(\beta - \tau)], \quad \tau \in (0, \beta). \tag{77}$$