EPR, Bohm and Bell

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1 Introduction

We discuss the most simple example for a so-called Bell-test experiment, where the spin state two spin-1/2 particles is prepared in the single-state of total spin $S = 0$ following [ST93]. Another very lucid derivation can be found in [Wei15]. The Hilbert space is $\mathcal{H} \otimes \mathcal{H}$ with $\mathcal{H}$ the two-dimensional Hilbert space of spin 1/2. As a basis we choose the usual eigenstates of $\mathbf{s}_3$. For simplicity we work with $\vec{\boldsymbol{\sigma}} = 2\vec{\mathbf{s}}$. Using "natural units" with $\hbar = 1$, the eigenvalues of $\boldsymbol{\sigma}_3$ are ± 1, and we denote the corresponding eigenstates with $|\sigma_3\rangle$, $\sigma_3 \in \{1, -1\}$. The prepared two-particle spin state is the pure state $\boldsymbol{\rho}_{AB} = |\Psi\rangle\langle\Psi|$ with

$$|\Psi\rangle = \frac{1}{\sqrt{2}}(|1\rangle \otimes |-1\rangle - |-1\rangle \otimes |1\rangle) \equiv \frac{1}{\sqrt{2}}|1 \otimes (-1) - (-1) \otimes 1\rangle. \tag{1}$$

In the following we label the gets of the single particles by their places A and B (where "Alice" and "Bob" place there spin-measurement apparatuses). Note that A and B can be very far distant. Note that $\boldsymbol{\rho}_{AB}$ is a self-adjoint operator acting in $\mathcal{H} \otimes \mathcal{H}$.
Now the state of Alice's spin is given by the reduced statistical operator

$$\boldsymbol{\rho}_A = \mathrm{tr}_B\, \boldsymbol{\rho}_{AB} = \sum_{\sigma_{31},\sigma_{32},\sigma_{33}} |\sigma_{31}\rangle \langle \sigma_{31} \otimes \sigma_{32} |\boldsymbol{\rho}| \sigma_{33} \otimes \sigma_{32}\rangle \langle \sigma_{33}| = \frac{1}{2}\mathbb{1}. \tag{2}$$

Note that $\boldsymbol{\rho}_A$ acts in $\mathcal{H}$. The state of the spin measured at Bob's site is the same

$$\boldsymbol{\rho}_B = \mathrm{tr}_A\, \boldsymbol{\rho}_{AB} = \sum_{\sigma_{31},\sigma_{32},\sigma_{33}} |\sigma_{31}\rangle \langle \sigma_{32} \otimes \sigma_{31} |\boldsymbol{\rho}| \sigma_{32} \otimes \sigma_{33}\rangle \langle \sigma_{33}| = \frac{1}{2}\mathbb{1}. \tag{3}$$

Both observers simply have an ideally unpolarized spin, which has maximal von-Neumann entropy for a two-level system

$$S_A = -\mathrm{tr}(\boldsymbol{\rho}_A \ln \boldsymbol{\rho}_A) = \ln 2, \quad S_B = -\mathrm{tr}(\boldsymbol{\rho}_B \ln \boldsymbol{\rho}_B) = \ln 2, \tag{4}$$

while the state of the two-spin system is a pure state and thus has entropy

$$S_{AB} = -\mathrm{tr}(\boldsymbol{\rho}_{AB} \ln \boldsymbol{\rho}_{AB}) = 0, \tag{5}$$

i.e., it represents complete knowledge about the two-spin observables. So concerning the single spins the state represents "maximum uncertainty" although the two-spin state represents complete knowledge in the sense of quantum mechanics, where complete knowledge also means that about most observables one has only probabilistic information. The state $\boldsymbol{\rho}_{AB}$ represents what's known as a Bell state,

i.e., it's a pure state, representing complete knowledge about the two-spin state, while the reduced states ρ_A and ρ_B of the single spins have maximum entropy, i.e., represent minimal possible knowledge.
Of course, what's prepared with this state is the total spin being $S = 0$, where

$$\vec{\mathbf{S}} = \vec{\mathbf{s}} \otimes \mathbb{1} + \mathbb{1} \times \vec{\mathbf{s}}. \tag{6}$$

It's easy to show that

$$\vec{\mathbf{S}}^2 |\Psi\rangle = 0, \quad \vec{n} \cdot \vec{\mathbf{S}} |\Psi\rangle = 0, \tag{7}$$

where $\vec{n}$ is an arbitrary unit vector. It's a rare example of a special state, where the entire set of spin components take simultaneously a determined value although these observables are mutually incompatible since all the spin operators of course fulfill the usual su(2)-commutation relations

$$\left[\vec{\mathbf{s}}_j, \vec{\mathbf{s}}_k\right] = \mathrm{i}\epsilon_{jkl} \vec{\mathbf{s}}_l. \tag{8}$$

Now the surprising property of a Bell state is that, although the single spins are maximally uncertain, preparation in the Bell state ρ_{AB} implies strong correlations between joint measurements.
The most simple example is that Alice and Bob both measure the spin component in the same direction. It doesn't matter, which direction since the total spin is $S = 0$, i.e., it's a completely isotropic state. So for convenience we choose the direction as $\vec{e}_3$. The probatility that A finds the value σ_{3A} and B finds the value σ_{3B} is given by

$$P_{AB}(\sigma_{3A}, \sigma_{3B}) = \mathrm{tr}(|\sigma_{3A} \otimes \sigma_{3B}\rangle \langle \sigma_{3A} \otimes \sigma_{3B}| \rho_{AB} = |\langle \sigma_{3A} \otimes \sigma_{3B} | \Psi \rangle|^2. \tag{9}$$

The simple calculation gives

$$P_{AB}(1,-1) = P_{AB}(-1,1) = \frac{1}{2}, \quad P_{AB}(1,1) = P_{AB}(-1,-1) = 0. \tag{10}$$

There is a "100% anticorrelation" although the single spins are maximally uncertain! I.e., if A finds spin up, 1, then B must find spin down, (-1), and vice versa. In no case one finds both spins up or both spins down.
Now this simple case could be also found by a "classical probability experiment", where we mean an experiment within what Bell calls a "local realistic" theory, where realism means that all observables of the system always take determined values, which for some reason are unknown to the experimentalists beforehand and that's why they get "random outcomes". Of course such a 100% anticorrelation is easily possible although the outcome of each observer's experiment is maximally unknown. E.g., one can put a red and a green ball in closed boxes and give with 50% chance the one or the other box to A and the other one to B. The uncertainty for each observer, which color he will find, is maximal but there is 100% anti-correlation, i.e., if A finds the red ball, then B must find the green one and vice versa. In this sense the above 100% correlation is no surprise.
We just mention for clarification that by "locality" Bell means that the outcomes of the measurements are not influenced by each other by assumption. We do not go into the still controversial discussion about the compatibility of this type of locality with locality in the sense of relativity, i.e., in connection with Einstein causality. I am of the opinion that both notions of "locality" are compatible because of the very construction of the only successfull type relatistic quantum theory (QT), i.e., "local quantum field theory", where "locality" is meant as "locality of interactions", i.e., that local observables are represented by self-adjoint operators that commute if their space-time arguments are space-like separated.

One should note that the very same "local QFT" also implies that "particles" (i.e., (asymptotically) free single-particle Fock states) are neither strictly localizable, nor does it exclude strong correlations of parts of a quantum systems over large distances. E.g., the particles whose spin we describe with the above formalism, may be measured at arbitrarily far places and still show the 100% anticorrelation although the outcome of measurements on each single-particle spin is maximally uncertain. The original EPR paper [EPR35] is notorious for its imprecise philosophical formulation and did *not* (sic!) contain Einstein's true quibble about quantum theory. It was not so much a possible violation of "Einstein locality" than the fact that there can be these long-ranged correlations, making distant parts of a quantum system not separated, i.e., probabilities for the outcome of (local) measurements on the two far distant particles need not necessarily factorize. This "inseparability" is precisely the case for entangled and most pronounced for the maximally entangled Bell states [Ein48].

It is Bell's great achievement that he put the before "only philosophical question" whether the probabilities predicted by QT can as well reproduced by some "local realistic theory", often called hidden variable (HV) theories, i.e., the assumption that there are "hidden observables" always determining the values of any measureable quntity of the system, while the hidden variables' values are unknown for some reason, such that also the measurement outcomes on true observables become random, to the realm of scientific decidebility, and it can be done with the two-spin example defined above. The trick is to measure the two spins in different directions on sufficiently large "ensembles of equallally prepared two-spin systems".

We start with the description of the experiment within the **local realistic HV theory**. Each A and B measure spin components (or rather the components of $\vec{\sigma} = 2\vec{s}$ in three different directions $\vec{a}$, $\vec{b}$, and $\vec{c}$[1]. Within the HV theory one assumes that the measured spin components all take deterministic values, which are however unknown due to the unknown values of the HVs, i.e., when measuring in directions $\vec{a}$, $\vec{b}$, $\vec{c}$ the possible outcomes are α, β, and $\gamma \in \{1,-1\}$ for each of the measured particles. That means we have the following table with probabilities for the joint outcomes $(\alpha_1, \beta_1, \gamma_1)$ for particle 1 (Alice's measurement) and $(\alpha_2, \beta_2, \gamma_2)$ for particle 2 (Bob's measurement):

Probability	Spin 1 (Alice)	Spin 2 (Bob)
	$\alpha_1, \beta_1, \gamma_1$	$\alpha_2, \beta_2, \gamma_2$
p(1,1,1)	$+++$	$---$
p(1,1,-1)	$++-$	$--+$
p(1,-1,1)	$+-+$	$-+-$
p(-1,1,1)	$-++$	$+--$
p(1,-1,-1)	$+--$	$-++$
p(-1,1,-1)	$-+-$	$+-+$
p(-1,-1,1)	$--+$	$++-$
p(-1,-1,-1)	$---$	$+++$

Of course, because our local realistic theory must reproduce the predictions of the quantum theory for our singlet state ρ_{AB} one must assume that always $\alpha_2 = -\alpha_1$, $\beta_2 = -\beta_1$, and $\gamma_2 = -\gamma_1$. That's why we have only the above listed 8 possible outcomes!

As stated in the footnote above, what really can be measured is always only one component at each site: E.g., Alice chooses direction $\vec{a}$, i.e., what's really observable by doing experiments on a sufficiently large

[1]One should note that each observer can only measure in one direction at one particle, i.e., the direction magnetic field of a Stern-Gerlach experiment determines only the spin component in that direction, i.e., what's meant is that both, Alice and Bob, can choose freely among the directions when measurering each particle's spin, and the two-spin system is assumed to be preparable with certainty in the Bell state ρ_{AB}.

sample of an ensemble defined by the preparation in the Bell state ρ_{AB}, are only probabilities like

$$p(\vec{a},\alpha_1;\vec{b},\beta_2) = p(\alpha_1,\beta_2,1) + p(\alpha_1,\beta_2,-1) \tag{11}$$

etc.

Bell investigates now the expectation values of the observables $(\vec{n}_1 \cdot \vec{\sigma} \otimes \vec{n}_2 \cdot \vec{\sigma})$, with arbitrary unit vecctors $\vec{n}_1$ and $\vec{n}_2$, i.e., the correlation functions,

$$C(\vec{n}_1,\vec{n}_2) = \langle (\vec{n}_1 \cdot \vec{\sigma}) \otimes (\vec{n}_1 \cdot \vec{\sigma}) \rangle \,. \tag{12}$$

The crucial point is that under the assumption of a local realistic theory, as defined by the probabilities as explained above **Bell's inequality** [Bel64],

$$|C(\vec{a},\vec{b}) - C(\vec{a},\vec{c})| \leq 1 + C(\vec{b},\vec{c}) \quad \text{for local realistic theory!} \tag{13}$$

and that this inequality is **violated** by quantum theory when the two spins are prepared in a Bell state like our singlet state ρ_{AB} described above.

So let's first **prove (13) for the local realistic theory**. With the above assigned probabilities we have

$$\begin{aligned} C(\vec{a},\vec{b}) &= \left\langle \boldsymbol{\sigma}_1(\vec{a})\boldsymbol{\sigma}_2(\vec{b}) \right\rangle = -\left\langle \boldsymbol{\sigma}_1(\vec{a})\boldsymbol{\sigma}_1(\vec{b}) \right\rangle = -\sum_{\alpha,\beta,\gamma} p(\alpha,\beta,\gamma)\alpha\beta, \\ C(\vec{a},\vec{c}) &= \langle \boldsymbol{\sigma}_1(\vec{a})\boldsymbol{\sigma}_2(\vec{c}) \rangle = -\langle \boldsymbol{\sigma}_1(\vec{a})\boldsymbol{\sigma}_1(\vec{c}) \rangle = -\sum_{\alpha,\beta,\gamma} p(\alpha,\beta,\gamma)\alpha\gamma. \end{aligned} \tag{14}$$

Since $\alpha^2 = \beta^2 = \gamma^2 = +1$, because $\alpha,\beta,\gamma \in \{1,-1\}$, we have

$$\begin{aligned} C(\vec{a},\vec{b}) - C(\vec{a},\vec{c}) &= -\sum_{\alpha,\beta,\gamma} p(\alpha,\beta,\gamma)(\alpha\beta - \alpha\gamma) \\ &= -\sum_{\alpha,\beta,\gamma} p(\alpha,\beta,\gamma)(\alpha\beta - \alpha\beta^2\gamma) \\ &= -\sum_{\alpha,\beta,\gamma} p(\alpha,\beta,\gamma)\alpha\beta(1-\beta\gamma). \end{aligned} \tag{15}$$

With the triangle inequality and because of $|\alpha\beta| = 1$ and $1 - \beta\gamma \in \{0,2\}$, i.e., $1 - \beta\gamma \geq 0$ daraus

$$\left| C(\vec{a},\vec{b}) - C(\vec{a},\vec{c}) \right| \leq \sum_{\alpha,\beta,\gamma} p(\alpha,\beta,\gamma)|\alpha\beta||1-\beta\gamma| = \sum_{\alpha,\beta,\gamma} p(\alpha,\beta,\gamma)(1-\beta\gamma) = 1 + C(\vec{b},\vec{c}) \tag{16}$$

follows, an this is indeed Bell's inequality (13).

Now we want to evaluate the correlation function (12) for **quantum theory** with the two-spin system prepared in the Bell singlet state ρ_{AB}. For the single spin we can use the realization with the usual Pauli matrices in the $\boldsymbol{\sigma}_z$ basis; leading to

$$\vec{a} \cdot \hat{\vec{\sigma}} = \begin{pmatrix} a_3 & a_1 - \mathrm{i}a_2 \\ a_1 + \mathrm{i}a_2 & -a_3 \end{pmatrix}, \tag{17}$$

where the matrix elements are simply given by

$$(\vec{a} \cdot \vec{\sigma})_{\sigma_1\sigma_2} = \langle \sigma_1 | \vec{a} \cdot \vec{\boldsymbol{\sigma}} | \sigma_2 \rangle \,. \tag{18}$$

Since for our pure state $\rho_{AB} = |\Psi\rangle\langle\Psi|$ the evaluation of the expectation value for any two-spin observable A simplifies to

$$\langle \mathbf{A} \rangle = \langle \Psi | \mathbf{A} | \Psi \rangle , \tag{19}$$

we get with $|\Psi\rangle$ given by (1)

$$\begin{aligned} C(\vec{a},\vec{b}) &= \frac{1}{2}\left\langle 1\otimes(-1)-(-1)\otimes 1 \middle| (\vec{a}\cdot\vec{\sigma})\otimes(\vec{b}\cdot\vec{\sigma}) \middle| 1\otimes(-1)-(-1)\otimes 1 \right\rangle \\ &= \frac{1}{2}\left[-a_3 b_3 - (a_1 - \mathrm{i}a_2)(b_1 + \mathrm{i}b_2) - (a_1 + \mathrm{i}a_2)(b_1 - \mathrm{i}b_2) - a_3 b_3\right] \\ &= -\vec{a}\cdot\vec{b}. \end{aligned} \tag{20}$$

We obtain a **contradiction to Bell's inequality** (13), i.e., a **contradiction to *any* local realistic theory** if we choose the unit vectors,

$$\vec{a} = (1,0,0), \quad \vec{b} = (0,1,0) \quad \text{und} \quad \vec{c} = \frac{1}{\sqrt{2}}(1,1,0) \tag{21}$$

because then

$$C(\vec{a},\vec{b}) = 0, \quad C(\vec{a},\vec{c}) = C(\vec{b},\vec{c}) = -\frac{1}{\sqrt{2}} \simeq -0{,}707 \tag{22}$$

and thus

$$|C(\vec{a},\vec{b}) - C(\vec{a},\vec{c})| \overset{\text{QM}}{=} \frac{1}{\sqrt{2}} \simeq 0{,}707 \quad \text{und} \quad 1 + C(\vec{b},\vec{c}) = 1 - \frac{1}{\sqrt{2}} \simeq 0{,}293, \tag{23}$$

i.e. inded the Bell inequality (13) is clearly violated!

References

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