Schwarzschild metric in General Relativity

In this worksheet the Schwarzschild metric is used to generate the components of different tensors used in general relativity. The following expressions are calculated automatically by Maple, whereas for convenience only the non zero components are shown:

1. The covariant metric tensor
2. Its determinant
3. Both Christoffel symbols of first and second kind
4. Riemann tensor
5. Ricci tensor and Ricci scalar
6. Einstein tensor
7. Weyl tensor

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restart :
with(tensor) : with(DifferentialGeometry) :
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Prepare the metric

Here we define the coordinate system.

$coords := [t, r, \theta, \phi]:$

Tell how the metric looks like.

$g\_compts := array(symmetric, sparse, 1..4, 1..4):$

Fill the diagonal components of the Schwarzschild metric.

$g\_compts_{1,1} := e^{\nu(r)} : g\_compts_{2,2} := -e^{\lambda(r)} : g\_compts_{3,3} := -r^2 : g\_compts_{4,4} := -r^2 \sin(\theta)^2 :$

Finally create the metric tensor.

$g := create([-1, -1], eval(g\_compts))$

$$table\left(\left[compts = \begin{bmatrix} e^{\nu(r)} & 0 & 0 & 0 \\ 0 & -e^{\lambda(r)} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2\sin(\theta)^2 \end{bmatrix}, index\_char = [-1, -1]\right]\right) \quad (1.1)$$

Calculate all tensors

Now let Maple calculate everything. Of course it is also possible to do every step separately and change accordingly.

$tensorsGR(coords, g, contra\_metric, det\_met, C1, C2, Rm, Rc, R, G, C)$
$display\_allGR(coords, g, contra\_metric, det\_met, C1, C2, Rm, Rc, R, G, C)$

The coordinates variables are :

$$x1 = t$$

$$x2 = r$$

$$x3 = \theta$$

$$x4 = \phi$$

The Covariant Metric

non-zero components :

$$cov\_g11 = e^{\nu(r)}$$

$$cov\_g22 = -e^{\lambda(r)}$$

$$cov\_g33 = -r^2$$

$$cov\_g44 = -r^2 \sin(\theta)^2$$

Determinant of the covariant metric tensor :

$$detg = -e^{\nu(r)} e^{\lambda(r)} r^4 \sin(\theta)^2$$

The Contravariant Metric

non-zero components :

$$contra\_g11 = \frac{1}{e^{\nu(r)}}$$

$$contra\_g22 = -\frac{1}{e^{\lambda(r)}}$$

$$contra\_g33 = -\frac{1}{r^2}$$

$$contra\_g44 = -\frac{1}{r^2 \sin(\theta)^2}$$

The Christoffel Symbols of the First Kind

non-zero components :

$$[11,2] = -\frac{1}{2}\left(\frac{d}{dr}\nu(r)\right)e^{\nu(r)}$$

$$[12,1] = \frac{1}{2}\left(\frac{d}{dr}\nu(r)\right)e^{\nu(r)}$$

$$[22,2] = -\frac{1}{2}\left(\frac{d}{dr}\lambda(r)\right)e^{\lambda(r)}$$

$$[23,3] = -r$$

$$[24,4] = -r\sin(\theta)^2$$

$$[33,2] = r$$

$$[34,4] = -r^2\sin(\theta)\cos(\theta)$$

$$[44,2] = r\sin(\theta)^2$$

$$[44,3] = r^2\sin(\theta)\cos(\theta)$$

The Christoffel Symbols of the Second Kind

non-zero components :

$$\{1,12\} = \frac{1}{2}\frac{d}{dr}\nu(r)$$

$$\{2,11\} = \frac{1}{2}\frac{\left(\frac{d}{dr}\nu(r)\right)e^{\nu(r)}}{e^{\lambda(r)}}$$

$$\{2,22\} = \frac{1}{2}\frac{d}{dr}\lambda(r)$$

$$\{2,33\} = -\frac{r}{e^{\lambda(r)}}$$

$$\{2,44\} = -\frac{r\sin(\theta)^2}{e^{\lambda(r)}}$$

$$\{3,23\} = \frac{1}{r}$$

$$\{3,44\} = -\sin(\theta)\cos(\theta)$$

$$\{4,24\} = \frac{1}{r}$$

$$\{4,34\} = \frac{\cos(\theta)}{\sin(\theta)}$$

The Riemann Tensor

non-zero components :

$$R1212 = -\frac{1}{2}\left(\frac{d^2}{dr^2}\nu(r)\right)e^{\nu(r)} - \frac{1}{4}\left(\frac{d}{dr}\nu(r)\right)^2 e^{\nu(r)} + \frac{1}{4}\left(\frac{d}{dr}\nu(r)\right)e^{\nu(r)}\left(\frac{d}{dr}\lambda(r)\right)$$

$$R1313 = -\frac{1}{2}\frac{\left(\frac{d}{dr}\nu(r)\right)e^{\nu(r)}\,r}{e^{\lambda(r)}}$$

$$R1414 = -\frac{1}{2}\frac{\left(\frac{d}{dr}\nu(r)\right)e^{\nu(r)}\,r\sin(\theta)^2}{e^{\lambda(r)}}$$

$$R2323 = -\frac{1}{2}\left(\frac{d}{dr}\lambda(r)\right)r$$

$$R2424 = -\frac{1}{2}\left(\frac{d}{dr}\lambda(r)\right)r\sin(\theta)^2$$

$$R3434 = \frac{r^2\left(\cos(\theta)^2 e^{\lambda(r)} - e^{\lambda(r)} + 1 - \cos(\theta)^2\right)}{e^{\lambda(r)}}$$

character : [-1, -1, -1, -1]

The Ricci tensor

non-zero components :

$$R11 = -\frac{1}{4}\frac{1}{r\,e^{\lambda(r)}}\left(e^{\nu(r)}\left(2\,r\left(\frac{d^2}{dr^2}\nu(r)\right) + r\left(\frac{d}{dr}\nu(r)\right)^2\right.\right.$$

$$- r \left(\frac{d}{dr}\nu(r)\right)\left(\frac{d}{dr}\lambda(r)\right) + 4\left(\frac{d}{dr}\nu(r)\right)\Bigg)\Bigg)$$

R22

$$= \frac{1}{4}\,\frac{1}{r}\left(2\,r\left(\frac{d^2}{dr^2}\nu(r)\right) + r\left(\frac{d}{dr}\nu(r)\right)^2 - r\left(\frac{d}{dr}\nu(r)\right)\left(\frac{d}{dr}\lambda(r)\right) - 4\left(\frac{d}{dr}\lambda(r)\right)\right)$$

$$R33 = -\frac{1}{2}\,\frac{\left(\frac{d}{dr}\lambda(r)\right) r + 2\,e^{\lambda(r)} - \left(\frac{d}{dr}\nu(r)\right) r - 2}{e^{\lambda(r)}}$$

$$R44 = \frac{1}{2}\,\frac{1}{e^{\lambda(r)}}\Bigg(\left(\frac{d}{dr}\nu(r)\right) r - \left(\frac{d}{dr}\nu(r)\right) r\cos(\theta)^2 - \left(\frac{d}{dr}\lambda(r)\right) r + \left(\frac{d}{dr}\lambda(r)\right) r\cos(\theta)^2 + 2\cos(\theta)^2\,e^{\lambda(r)} - 2\,e^{\lambda(r)} + 2 - 2\cos(\theta)^2\Bigg)$$

character : [-1, -1]

The Ricci Scalar

$$R = \frac{1}{2}\,\frac{1}{r^2\,e^{\lambda(r)}}\left(-2\,r^2\left(\frac{d^2}{dr^2}\nu(r)\right) - r^2\left(\frac{d}{dr}\nu(r)\right)^2 + r^2\left(\frac{d}{dr}\nu(r)\right)\left(\frac{d}{dr}\lambda(r)\right) - 4\left(\frac{d}{dr}\nu(r)\right) r + 4\left(\frac{d}{dr}\lambda(r)\right) r + 4\,e^{\lambda(r)} - 4\right)$$

The Einstein Tensor

non-zero components :

$$G11 = -\frac{e^{\nu(r)}\left(\left(\frac{d}{dr}\lambda(r)\right) r + e^{\lambda(r)} - 1\right)}{r^2\,e^{\lambda(r)}}$$

$$G22 = \frac{-\left(\frac{d}{dr}\nu(r)\right) r + e^{\lambda(r)} - 1}{r^2}$$

$$G33=-\frac{1}{4}\,\frac{1}{e^{\lambda(r)}}\left(r\left(-2\left(\frac{d}{dr}\lambda(r)\right)+2\left(\frac{d}{dr}\nu(r)\right)+2\,r\left(\frac{d^2}{dr^2}\nu(r)\right)+r\left(\frac{d}{dr}\nu(r)\right)^2-r\left(\frac{d}{dr}\nu(r)\right)\left(\frac{d}{dr}\lambda(r)\right)\right)\right)$$

$$G44=\frac{1}{4}\,\frac{1}{e^{\lambda(r)}}\left(r\left(-2\left(\frac{d}{dr}\nu(r)\right)+2\left(\frac{d}{dr}\nu(r)\right)\cos(\theta)^2+2\left(\frac{d}{dr}\lambda(r)\right)-2\left(\frac{d}{dr}\lambda(r)\right)\cos(\theta)^2-2\,r\left(\frac{d^2}{dr^2}\nu(r)\right)+2\,r\left(\frac{d^2}{dr^2}\nu(r)\right)\cos(\theta)^2-r\left(\frac{d}{dr}\nu(r)\right)^2+r\left(\frac{d}{dr}\nu(r)\right)^2\cos(\theta)^2+r\left(\frac{d}{dr}\nu(r)\right)\left(\frac{d}{dr}\lambda(r)\right)-r\left(\frac{d}{dr}\nu(r)\right)\left(\frac{d}{dr}\lambda(r)\right)\cos(\theta)^2\right)\right)$$

character : [-1, -1]

The Weyl Tensor

non-zero components :

$$C1212=\frac{1}{12}\,\frac{1}{r^2}\left(e^{\nu(r)}\left(-2\,r^2\left(\frac{d^2}{dr^2}\nu(r)\right)-r^2\left(\frac{d}{dr}\nu(r)\right)^2+r^2\left(\frac{d}{dr}\nu(r)\right)\left(\frac{d}{dr}\lambda(r)\right)-2\left(\frac{d}{dr}\lambda(r)\right)r+2\left(\frac{d}{dr}\nu(r)\right)r+4\,e^{\lambda(r)}-4\right)\right)$$

$$C1313=-\frac{1}{24}\,\frac{1}{e^{\lambda(r)}}\left(e^{\nu(r)}\left(-2\,r^2\left(\frac{d^2}{dr^2}\nu(r)\right)-r^2\left(\frac{d}{dr}\nu(r)\right)^2+r^2\left(\frac{d}{dr}\nu(r)\right)\left(\frac{d}{dr}\lambda(r)\right)-2\left(\frac{d}{dr}\lambda(r)\right)r+2\left(\frac{d}{dr}\nu(r)\right)r+4\,e^{\lambda(r)}-4\right)\right)$$

$$C1414=\frac{1}{24}\,\frac{1}{e^{\lambda(r)}}\left(e^{\nu(r)}\left(-2\left(\frac{d}{dr}\nu(r)\right)r+2\left(\frac{d}{dr}\nu(r)\right)r\cos(\theta)^2+2\left(\frac{d}{dr}\lambda(r)\right)r-2\left(\frac{d}{dr}\lambda(r)\right)r\cos(\theta)^2+4\cos(\theta)^2\,e^{\lambda(r)}-4\,e^{\lambda(r)}+4-4\cos(\theta)^2+2\,r^2\left(\frac{d^2}{dr^2}\nu(r)\right)-2\,r^2\left(\frac{d^2}{dr^2}\nu(r)\right)\cos(\theta)^2+r^2\left(\frac{d}{dr}\nu(r)\right)^2\right.\right.$$

$$- r^2 \left(\frac{d}{dr} \nu(r)\right)^2 \cos(\theta)^2 - r^2 \left(\frac{d}{dr} \nu(r)\right) \left(\frac{d}{dr} \lambda(r)\right)$$

$$+ r^2 \left(\frac{d}{dr} \nu(r)\right) \left(\frac{d}{dr} \lambda(r)\right) \cos(\theta)^2 \Big) \Big)$$

$$C2323 = - \frac{1}{12} \left(\frac{d}{dr} \lambda(r)\right) r + \frac{1}{6} e^{\lambda(r)} + \frac{1}{12} \left(\frac{d}{dr} \nu(r)\right) r - \frac{1}{6}$$

$$- \frac{1}{12} r^2 \left(\frac{d^2}{dr^2} \nu(r)\right) - \frac{1}{24} r^2 \left(\frac{d}{dr} \nu(r)\right)^2 + \frac{1}{24} r^2 \left(\frac{d}{dr} \nu(r)\right) \left(\frac{d}{dr} \lambda(r)\right)$$

$$C2424 = \frac{1}{12} \left(\frac{d}{dr} \nu(r)\right) r - \frac{1}{12} \left(\frac{d}{dr} \nu(r)\right) r \cos(\theta)^2 - \frac{1}{12} \left(\frac{d}{dr} \lambda(r)\right) r$$

$$+ \frac{1}{12} \left(\frac{d}{dr} \lambda(r)\right) r \cos(\theta)^2 - \frac{1}{6} \cos(\theta)^2 e^{\lambda(r)} + \frac{1}{6} e^{\lambda(r)} - \frac{1}{6} + \frac{1}{6} \cos(\theta)^2$$

$$- \frac{1}{12} r^2 \left(\frac{d^2}{dr^2} \nu(r)\right) + \frac{1}{12} r^2 \left(\frac{d^2}{dr^2} \nu(r)\right) \cos(\theta)^2 - \frac{1}{24} r^2 \left(\frac{d}{dr} \nu(r)\right)^2$$

$$+ \frac{1}{24} r^2 \left(\frac{d}{dr} \nu(r)\right)^2 \cos(\theta)^2 + \frac{1}{24} r^2 \left(\frac{d}{dr} \nu(r)\right) \left(\frac{d}{dr} \lambda(r)\right)$$

$$- \frac{1}{24} r^2 \left(\frac{d}{dr} \nu(r)\right) \left(\frac{d}{dr} \lambda(r)\right) \cos(\theta)^2$$

$$C3434 = \frac{1}{12} \frac{1}{e^{\lambda(r)}} \Big(r^2 \Big(-2 \left(\frac{d}{dr} \nu(r)\right) r + 2 \left(\frac{d}{dr} \nu(r)\right) r \cos(\theta)^2 + 2 \left(\frac{d}{dr} \lambda(r)\right) r$$

$$- 2 \left(\frac{d}{dr} \lambda(r)\right) r \cos(\theta)^2 + 4 \cos(\theta)^2 e^{\lambda(r)} - 4 e^{\lambda(r)} + 4 - 4 \cos(\theta)^2$$

$$+ 2 r^2 \left(\frac{d^2}{dr^2} \nu(r)\right) - 2 r^2 \left(\frac{d^2}{dr^2} \nu(r)\right) \cos(\theta)^2 + r^2 \left(\frac{d}{dr} \nu(r)\right)^2$$

$$- r^2 \left(\frac{d}{dr} \nu(r)\right)^2 \cos(\theta)^2 - r^2 \left(\frac{d}{dr} \nu(r)\right) \left(\frac{d}{dr} \lambda(r)\right)$$

$$+ r^2 \left(\frac{d}{dr} \nu(r)\right) \left(\frac{d}{dr} \lambda(r)\right) \cos(\theta)^2 \Big) \Big)$$

character : [-1, -1, -1, -1]

(2.1)