Merging Neutron Stars

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WE-Heraeus summer school on “Nuclear Physics in Astrophysics”
Max Planck Institute for Nuclear Physics (MPIK), Heidelberg , 10-14.09.19

GOETHE UNIVERSITÄT FRANKFURT AM MAIN

erc

MAX-PLANCK-INSTITUT FÜR KERNPHYSIK HEIDELBERG

WILHELM UND ELSE HERAEUS-STIFTUNG

Plan of the lectures

- Lecture I: the **math** of neutron-star mergers

- Lecture II: the **physics/astrophysics** of neutron-star mergers

- Alcubierre, ***"Introduction to 3+1 Numerical Relativity"***, Oxford University Press, 2008
- Baumgarte and Shapiro, ***"Numerical Relativity: Solving Einstein's Equations on the Computer"***, Cambridge University Press, 2010
- Gourgoulhon, "***3+1 Formalism in General Relativity",*** Lecture Notes in Physics, Springer 2012
- Rezzolla and Zanotti, "Relativistic Hydrodynamics", Oxford University Press, 2013

Merging Neutron Stars

- <u>Lecture I: The math of neutron-star mergers</u>
  - Introduction
  - A brief review of General Relativity
  - Numerical relativity of neutron-star mergers
    - The 3+1 decomposition of spacetime
    - ADM equations
    - BSSNOK/ccZ4 formulation
    - Initial data, gauge conditions, excising parts of spacetime and gravitational wave extraction
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  - Introduction
  - GW170817 - the long-awaited event
  - Determining neutron-star properties and the equation of state using gravitational wave data
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  - Detecting the hadron-quark phase transition with gravitational waves

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Binary Neutron Star Systems

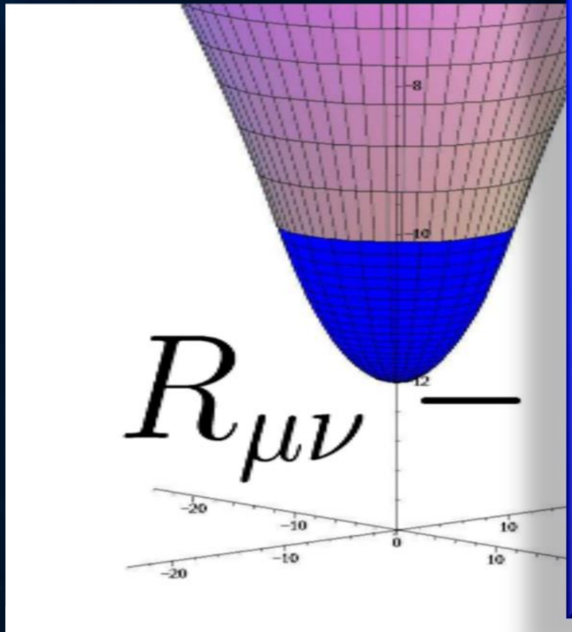
Kramer, Wex, Class. Quantum Grav. 2009

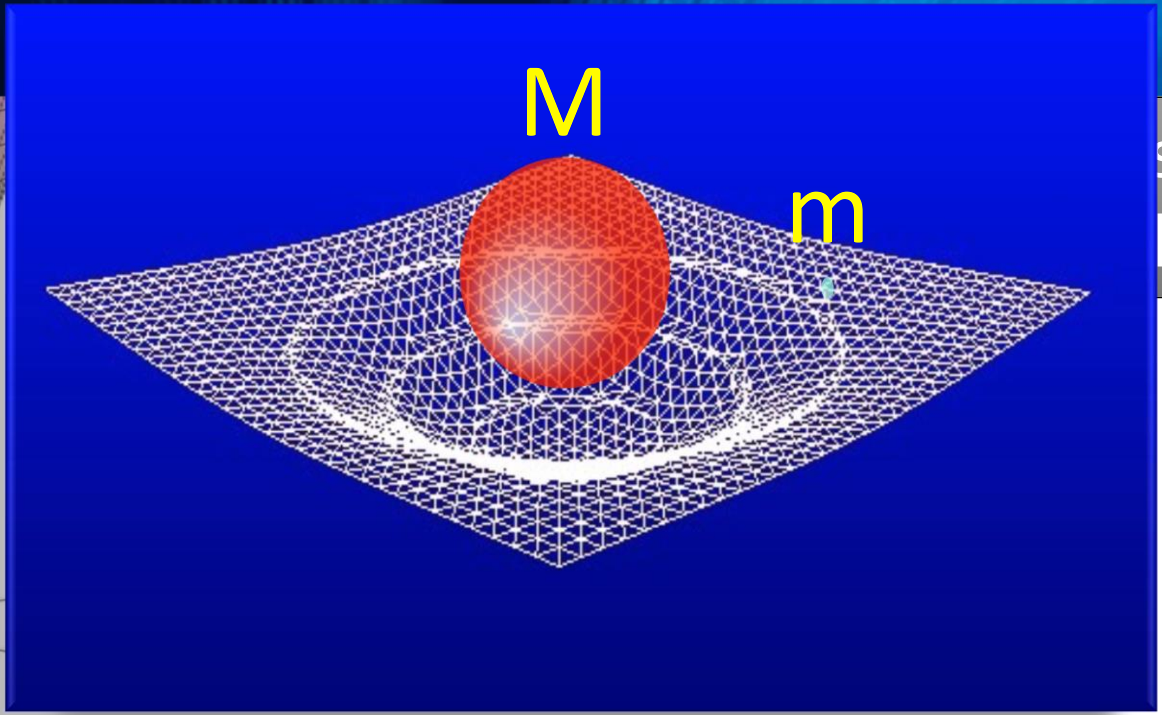
The **Double Pulsar** (PSR J0737-3039A/B):
Observed in 2003
Eccentricity: 0.088
Pulsar A: P=23 ms, M=1.3381(7)
Pulsar B: P=2.7 s, M=1.2489(7)
Only separated 800,000 km from each other
Orbital period: 147 Minuten
Pulsar A is eclipsed by Pulsar B
(30 s for each orbit)

Distance shrinks
due to Gravitational Wave emission
→ They will collide in 85 Million Years!

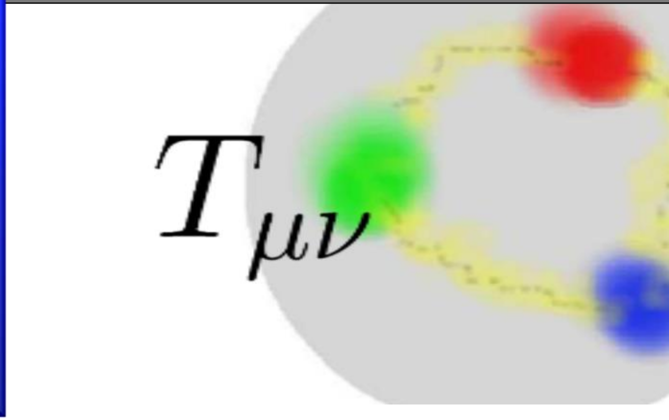
McGill NCS Multimedia Services Animation by Daniel Cantin, DarwinDimensions)

General Relativity
The Einstein equation





stein presented his
l Relativity " (GR)



Curvature of Spacetime = Energy

GR has been a revolutionary theory in physics: Sie besagt, dass jegliche Energieformen (z.B. Masse der Erde) die „Raumzeit" verbiegen und durch diese Krümmung des Raumes und der Zeit resultiert die Gravitationkraft (Schwerkraft).

Gravitative Zeitdilatation

Den Effekt der Zeitverbiegung kann man heutzutage sogar auf der Erde nachweisen -> Uhren ticken in den Bergen ein wenig schneller als im Tal.

News
12.02.2018
Drucken
Teilen

RELATIVITÄTSTHEORIE

Warum die Zeit im Gebirge schneller vergeht

Mit einem surrealen Effekt der Gravitationsphysik haben Wissenschaftler die Höhe eines Tunnels in den französischen Alpen bestimmt.

von Robert Gast

© ISTOCK / SKOUATROULIO (AUSSCHNITT)

2018 auf www.spektrum.de

Frankfurter Allgemeine
Physik & Mehr

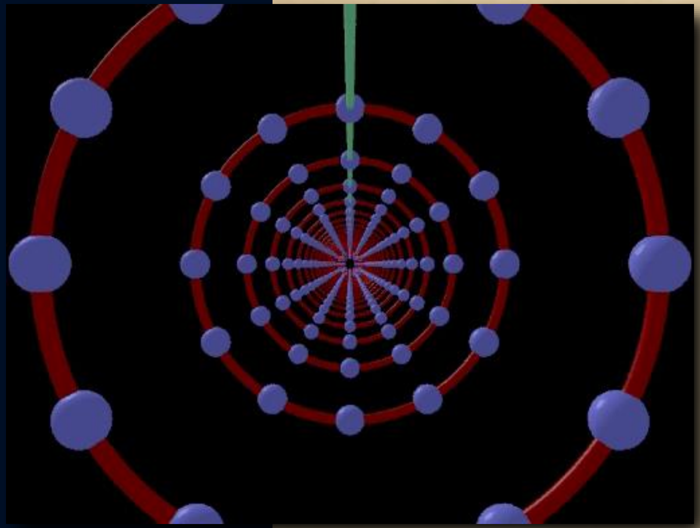
DIZIN GENE KLIMA WELTRAUM GARTEN NETZRÄTSEL

ALLGEMEINE RELATIVITÄTSTHEORIE

Hurra, wir hier unten leben länger!

VON ANNE HARDY - AKTUALISIERT AM 19.10.2010 - 06:00

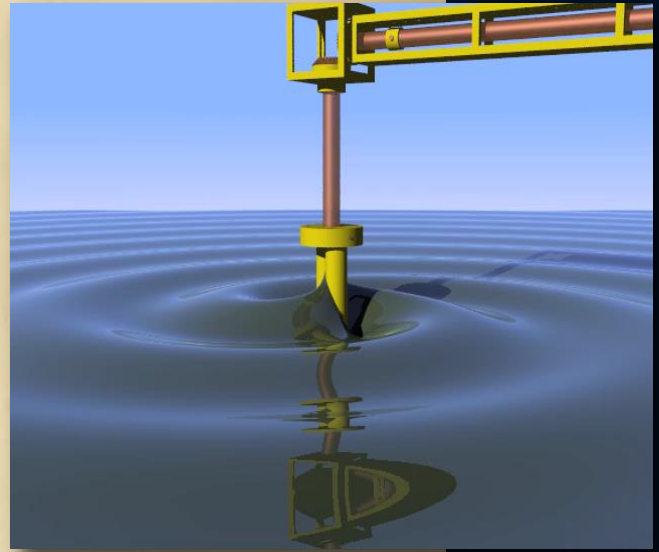
154 Gesamtsitzung vom 14. Februar 1918. — Mitteilung vom 31. Januar

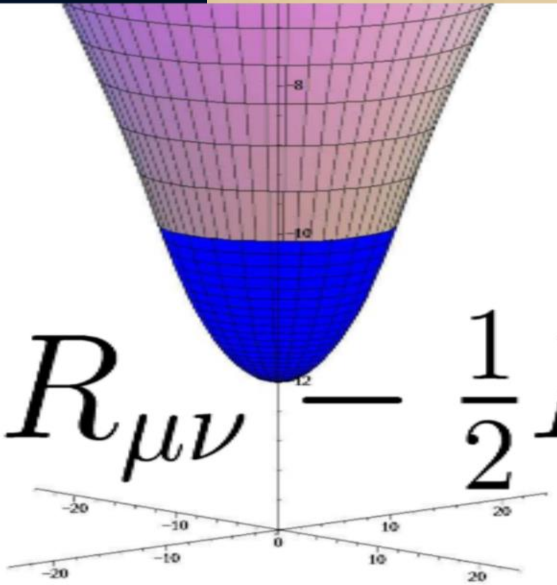


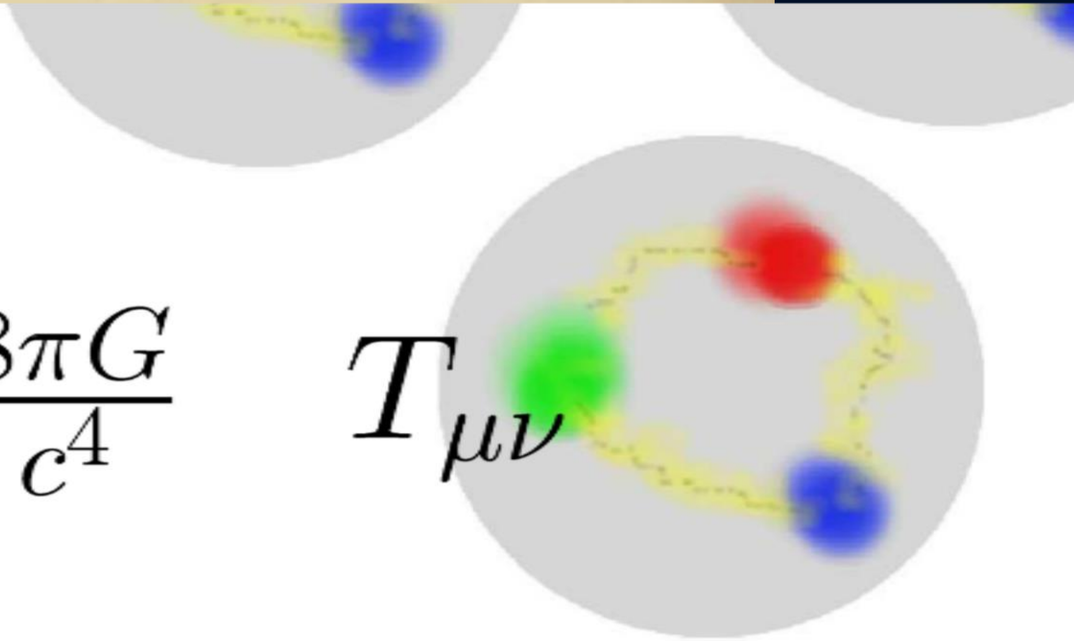
Über Gravitationswellen.

Von A. Einstein.

(Vorgelegt am 31. Januar 1918 [s. oben S. 79].)







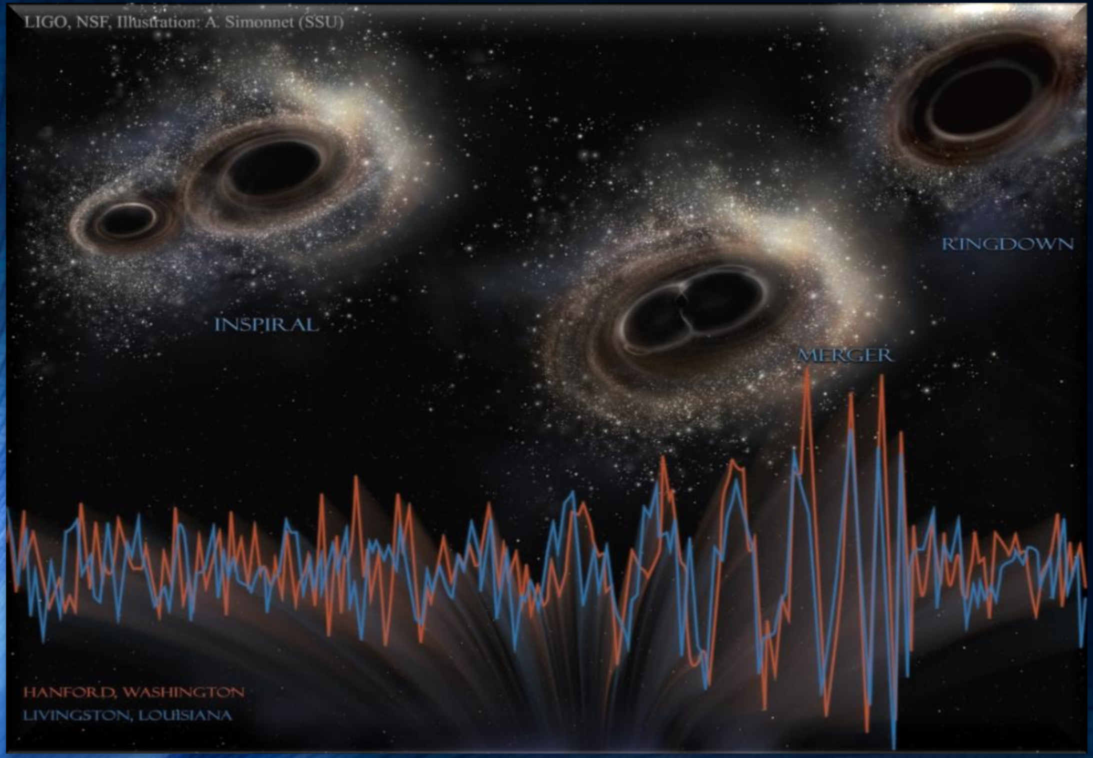
$$R_{\mu\nu} - \frac{1}{2} R \, g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

Erste Gravitationswelle im Jahr 2015 gefunden!!

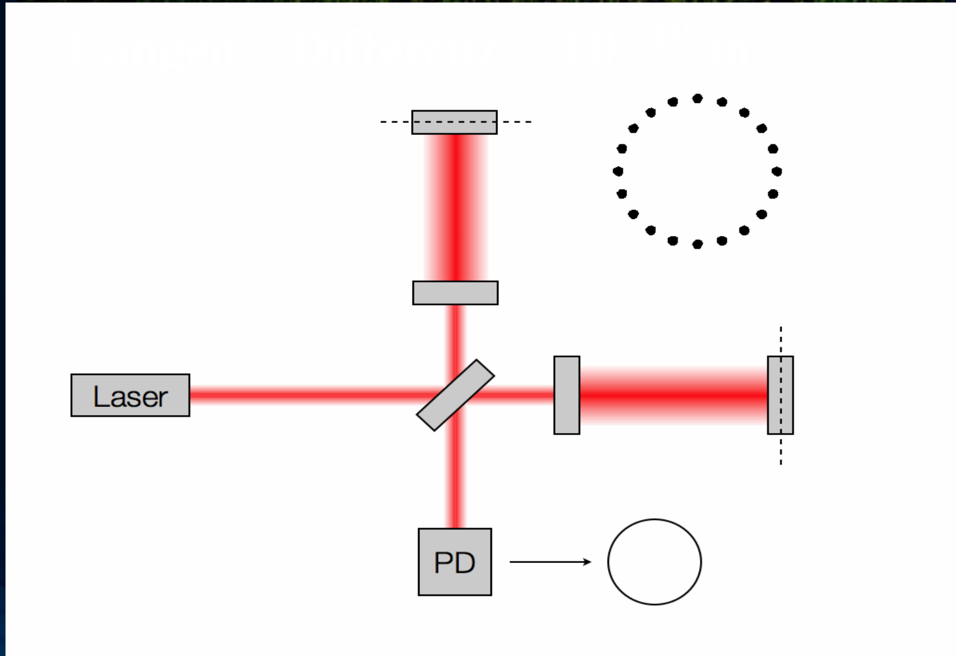
Kollision zweier Schwarzer Löcher GW150914

Massen: 36 & 29 Sonnenmassen

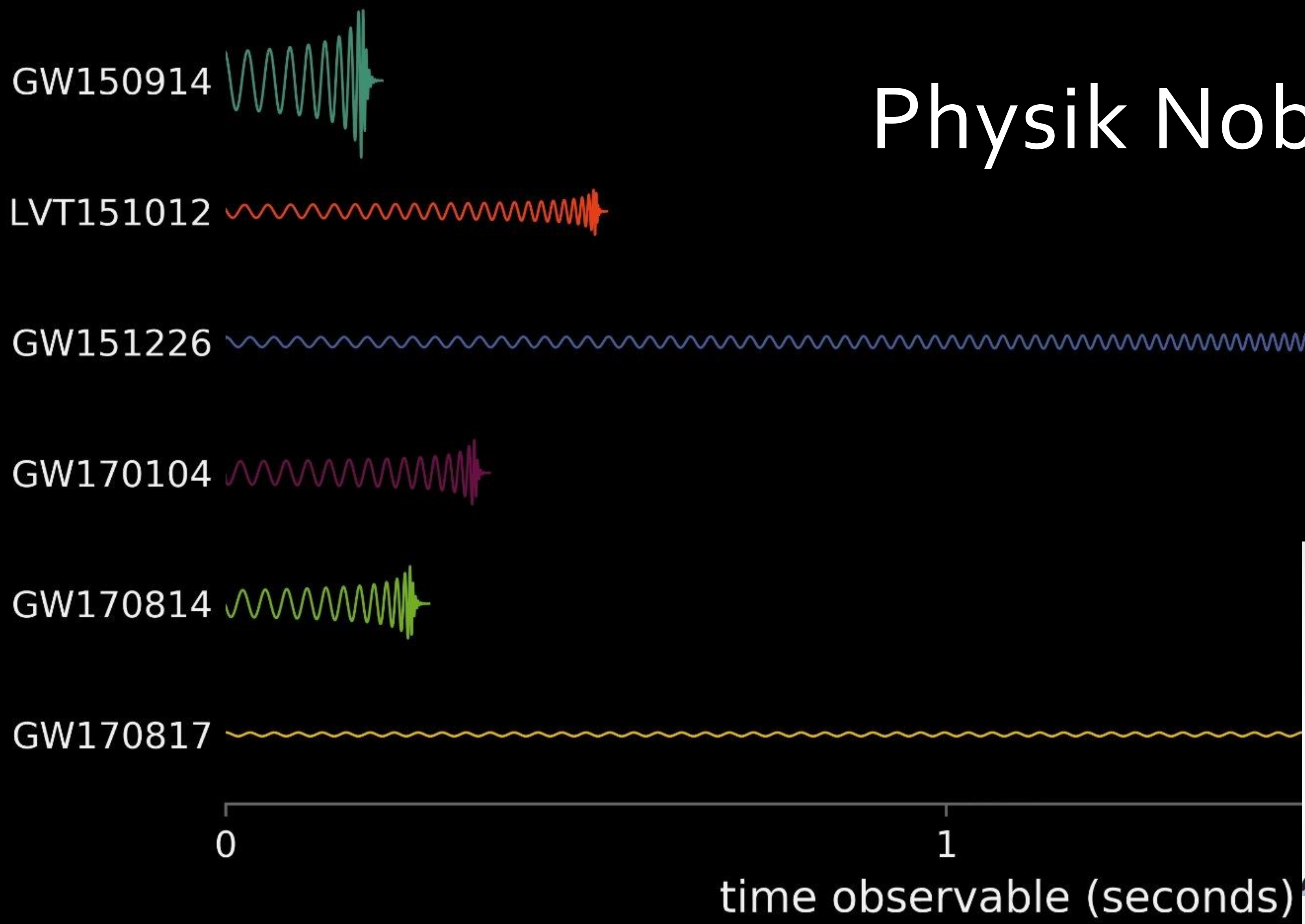
Abstand zur Erde 410 Mpc (1.34 Milliarden Lichtjahre)

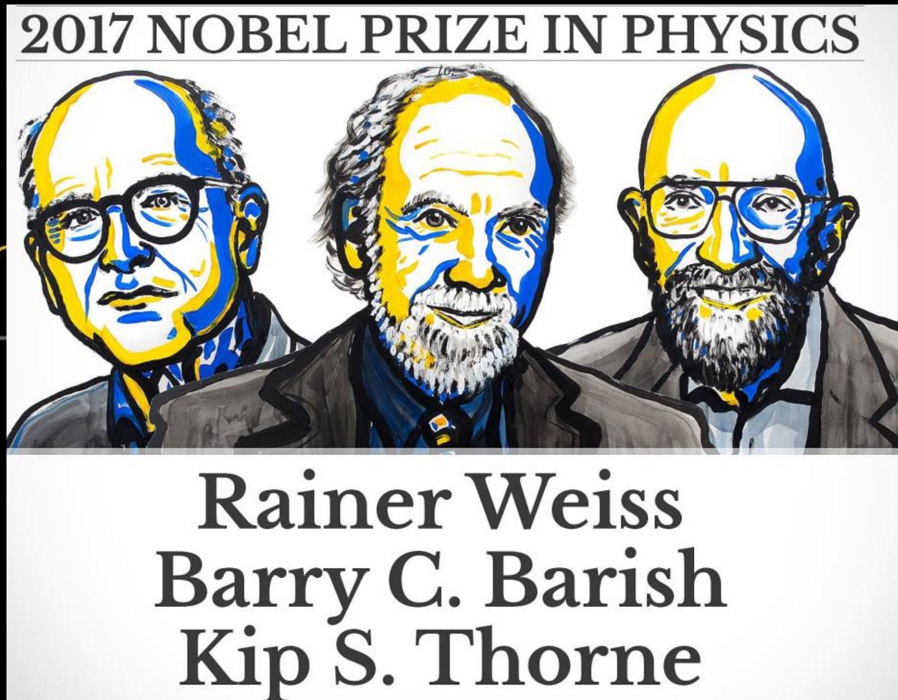


LIGO Gravitationswellen Detektor



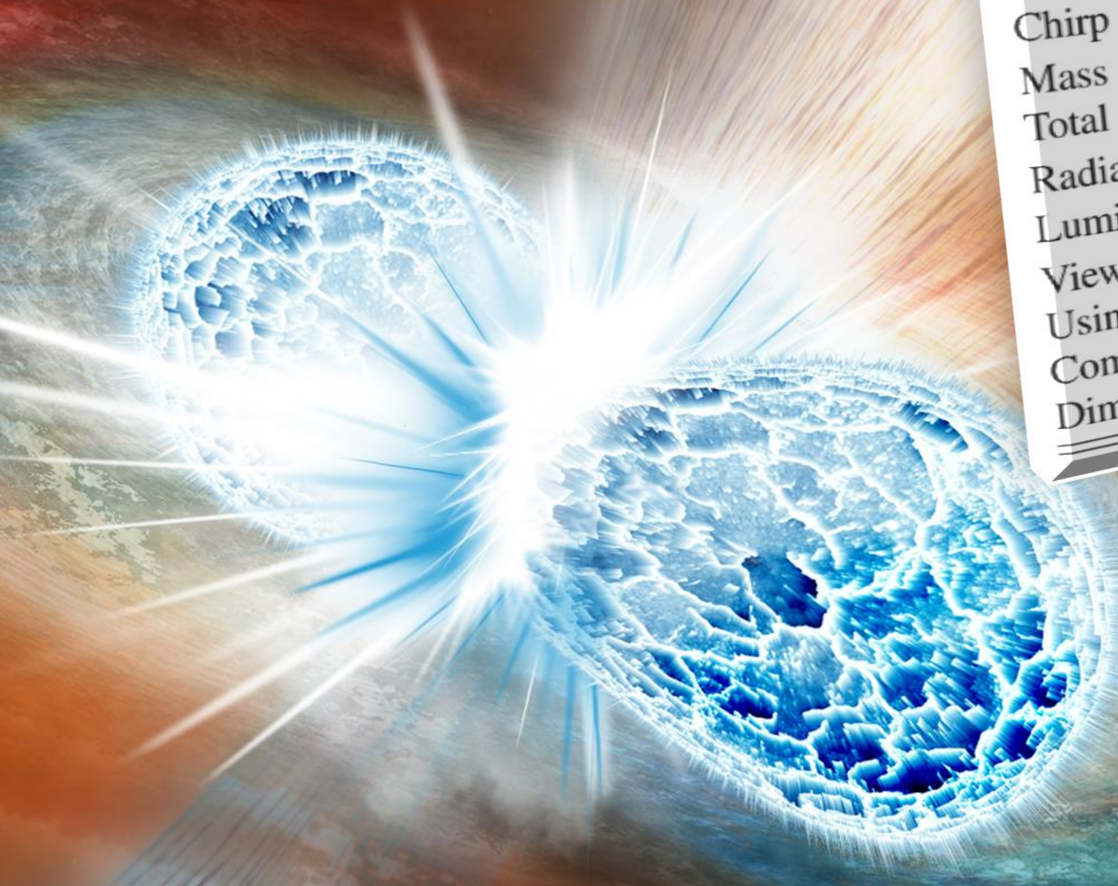
Physik Nobel Preis 2017





The long awaited event GW170817

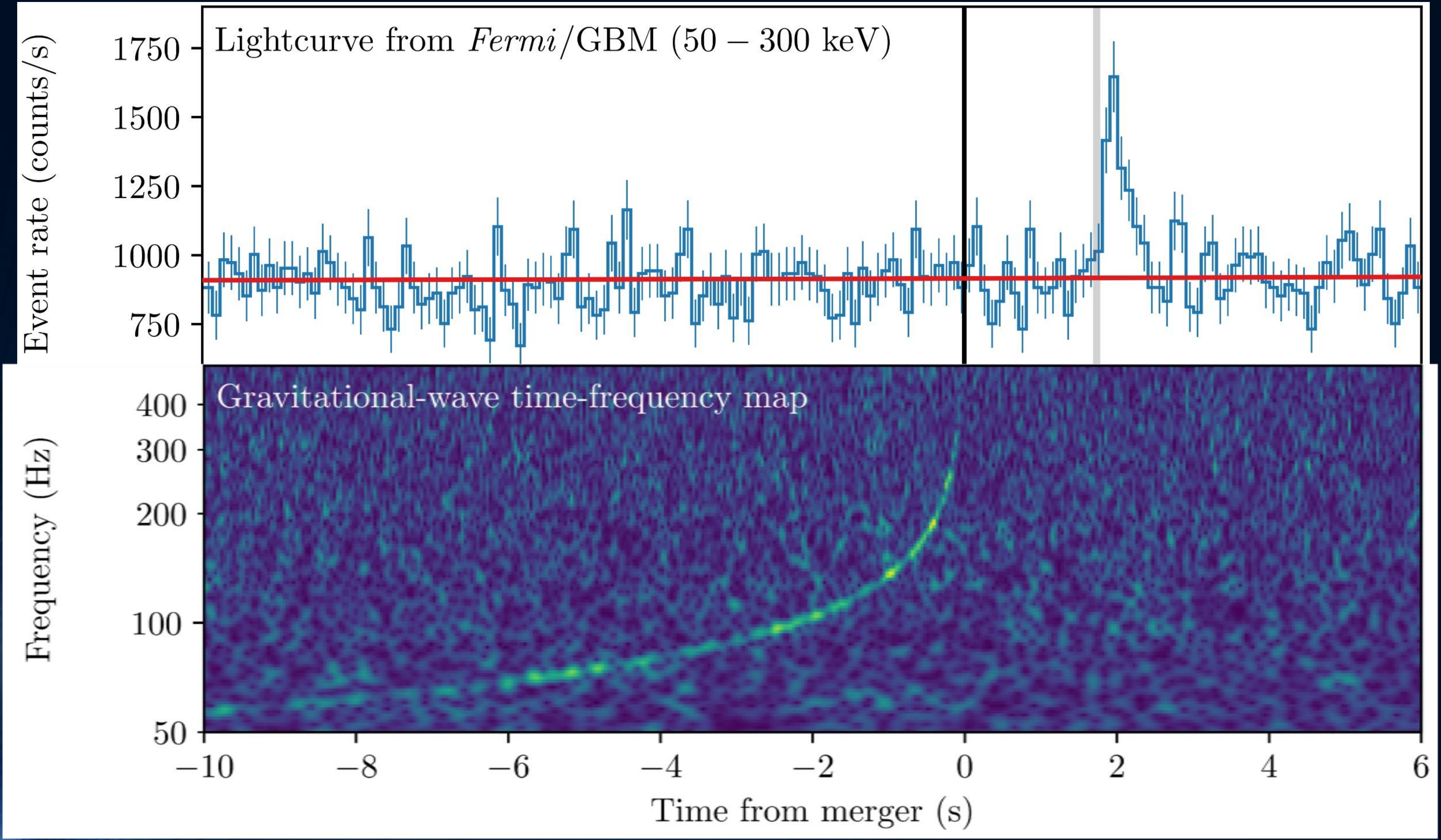
	Low-spin priors ($\|\chi\| \leq 0.05$)	High-spin priors ($\|\chi\| \leq 0.89$)
Primary mass m_1	$1.36–1.60\ M_\odot$	$1.36–2.26\ M_\odot$
Secondary mass m_2	$1.17–1.36\ M_\odot$	$0.86–1.36\ M_\odot$
Chirp mass $\mathcal{M}$	$1.188^{+0.004}_{-0.002} M_\odot$	$1.188^{+0.004}_{-0.002} M_\odot$
Mass ratio m_2/m_1	$0.7–1.0$	$0.4–1.0$
Total mass $m_{\rm tot}$	$2.74^{+0.04}_{-0.01} M_\odot$	$2.82^{+0.47}_{-0.09} M_\odot$
Radiated energy $E_{\rm rad}$	$> 0.025 M_\odot c^2$	$> 0.025 M_\odot c^2$
Luminosity distance $D_{\rm L}$	40^{+8}_{-14} Mpc	40^{+8}_{-14} Mpc
Viewing angle Θ	$\lesssim 55°$	$\lesssim 56°$
Using NGC 4993 location	$\lesssim 28°$	$\lesssim 28°$
Combined dimensionless tidal deformability $\tilde{\Lambda}$	$\lesssim 800$	$\lesssim 700$
Dimensionless tidal deformability $\Lambda(1.4 M_\odot)$	$\lesssim 800$	$\lesssim 1400$



17. August 2017

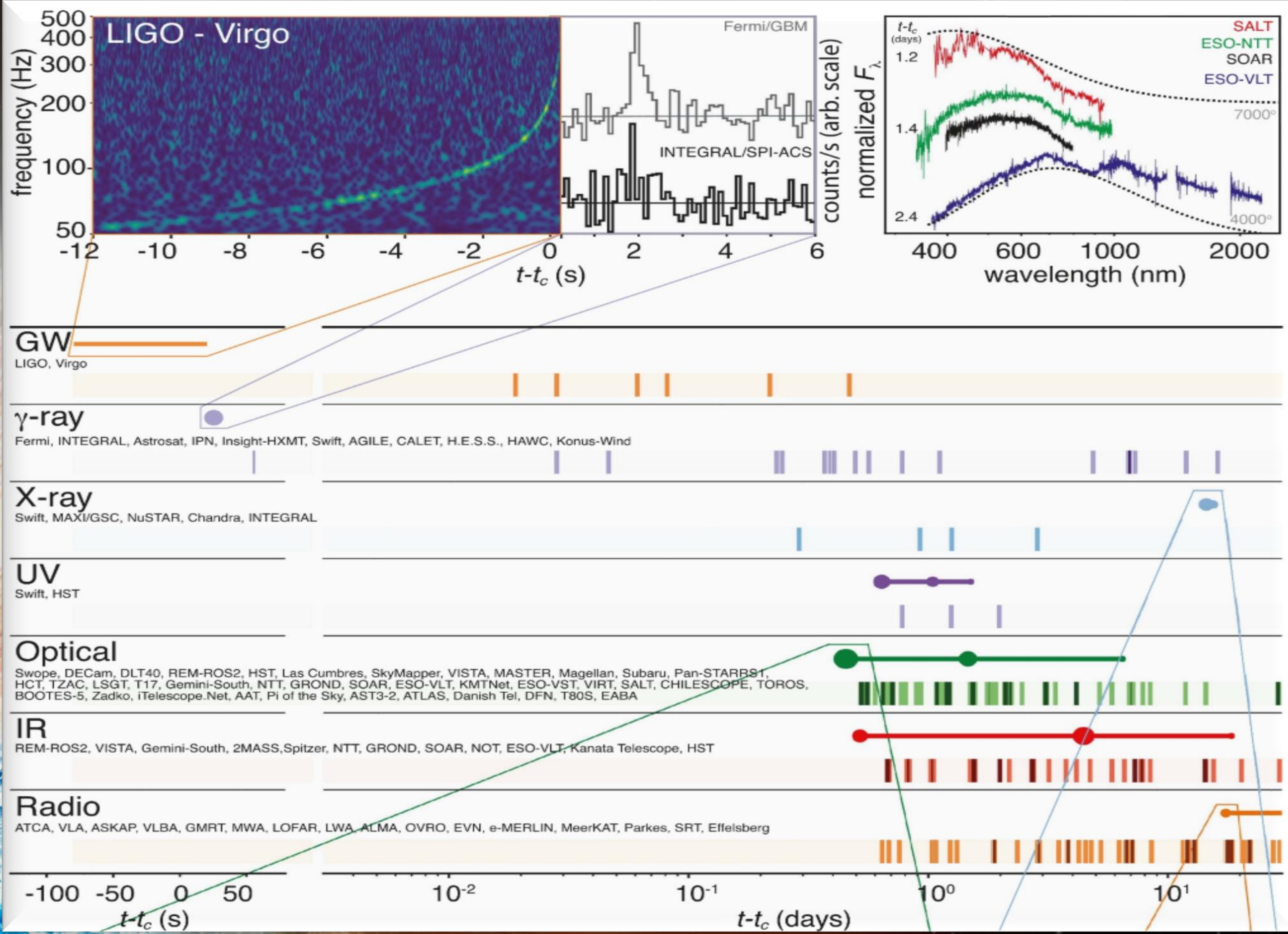
Gravitationswelle einer
Neutronenstern Kollision gemessen!

Gravitational Wave GW170817 and Gamma-Ray Emission GRB170817A



GW170817

Kilonova observed



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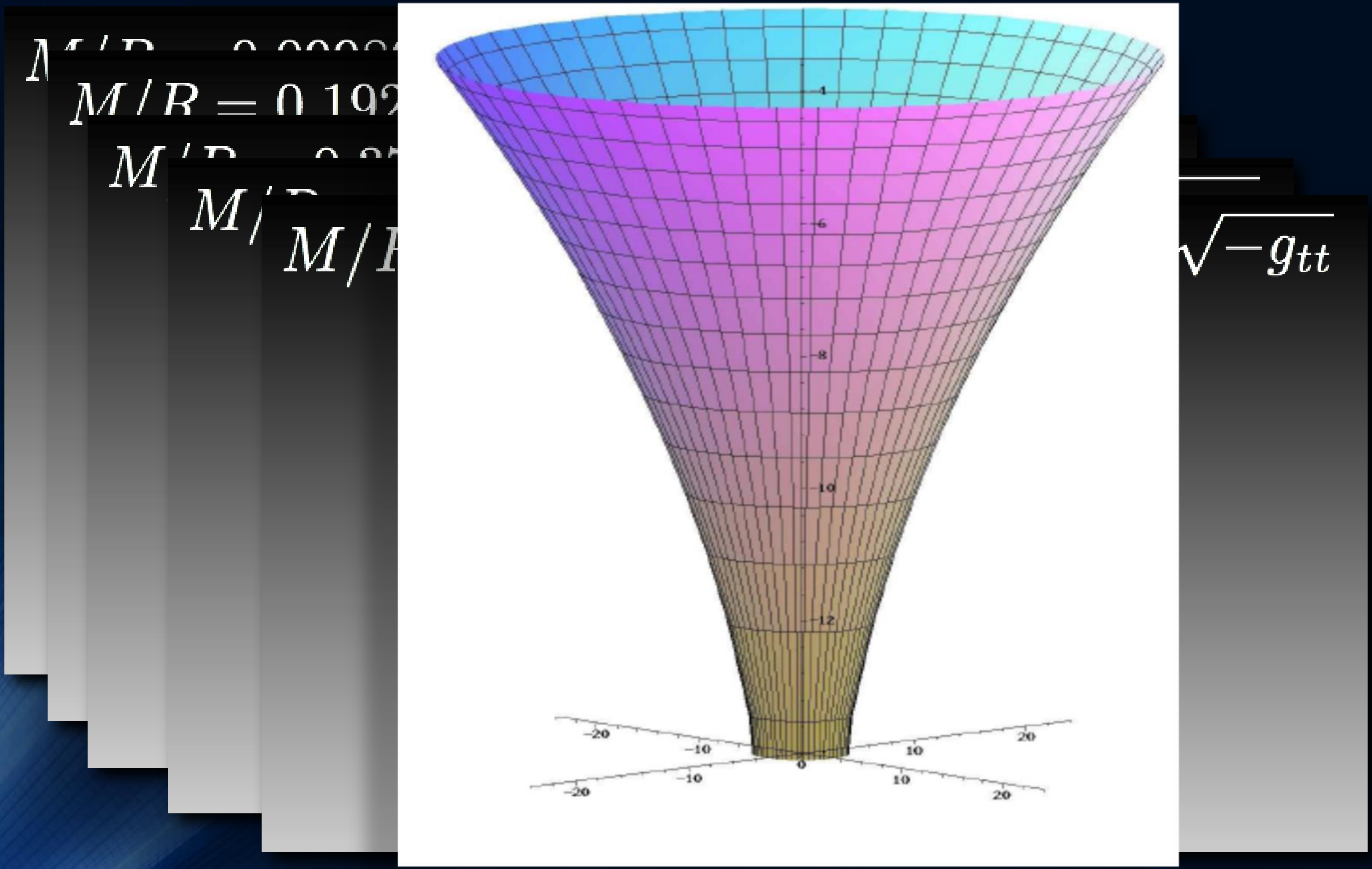
Die Schwarzschild Lösung

1915 Einsteins Gravitation:
Krümmung der „Raumzeit"

1916 Karl Schwarzschild:
… geboren 1873 in Frankfurt nahe dem Haus der Rothschild's. Erste Lösung der ART – drei Monate nach Einsteins Artikel! Aussenraummetrik eines nichtrotierenden schwarzen Loches.

Schwarzschild stirbt einen Monat später an einer Infektion die er sich an der russischen Front einfing…

Schwarze Löcher und der Raumzeit-Trichter



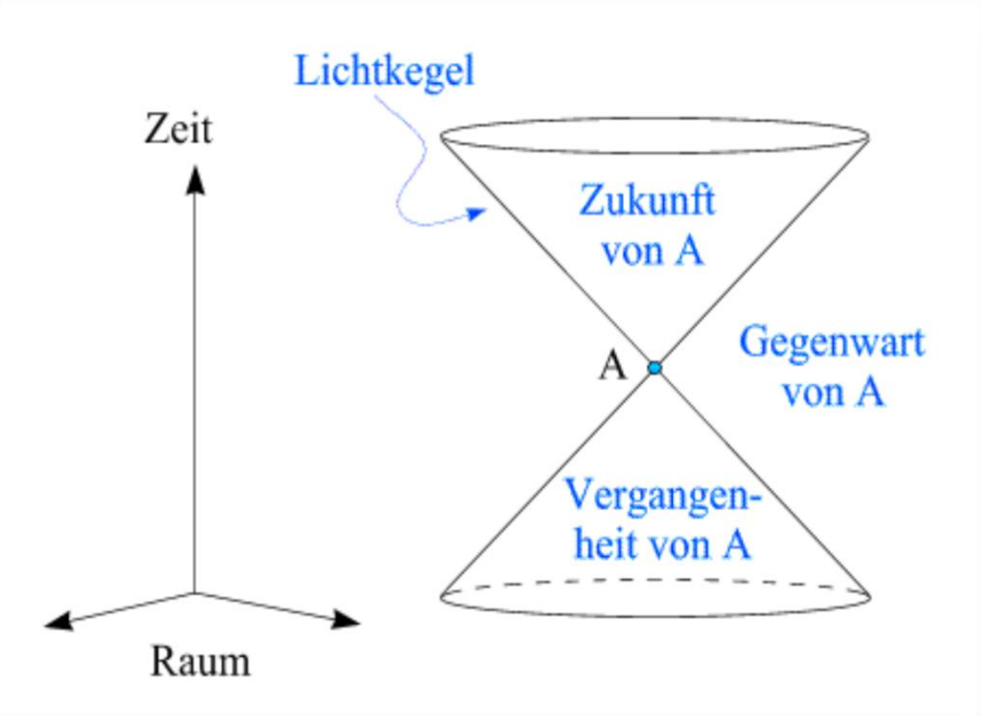
M: Masse des Objektes
R: Radius des Objektes
g_{tt}: Metrik der Raumzeit

Wir sind über den Grenzwert gekommen und haben ein schwarzes Loch erzeugt!

Grenzwert der Krümmung: Stabile Objekte (Neutronensterne) sind nicht mehr möglich

Raumzeit-Diagramm eines schwarzen Loches

Sichtweise ruhender Beobachter im Unendlichen



Raumzeit-Struktur
im flachen Raum

Ereignis-
horizont

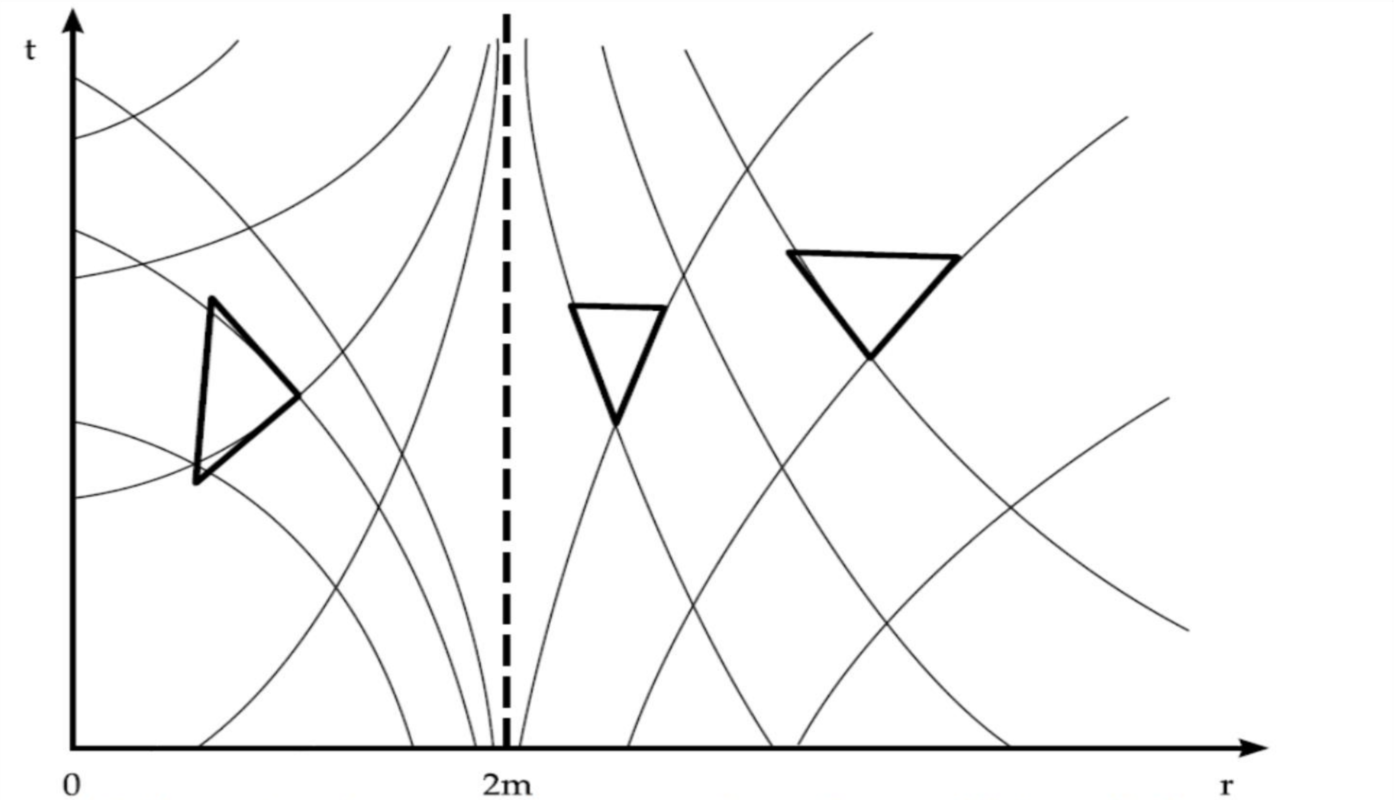


Abbildung 2.1: Raumzeitdiagramm der Schwarzschildmetrik in Schwarzschildkoordinaten.

Raumzeit-Struktur um ein schwarzes Loch

Raumzeit-Diagramm eines schwarzen Loches

Sichtweise eines in das schwarze Loch fallenden Beobachters

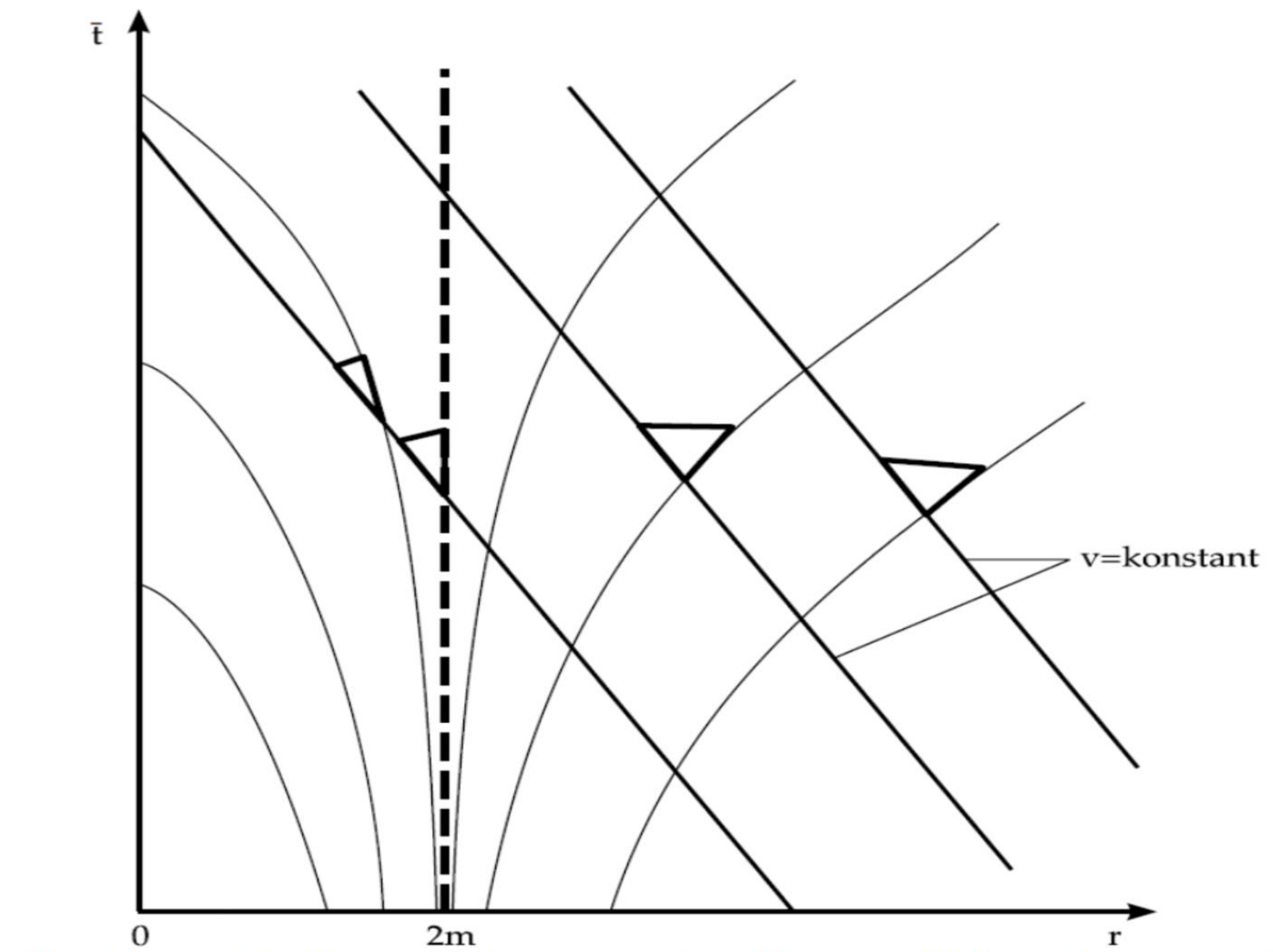
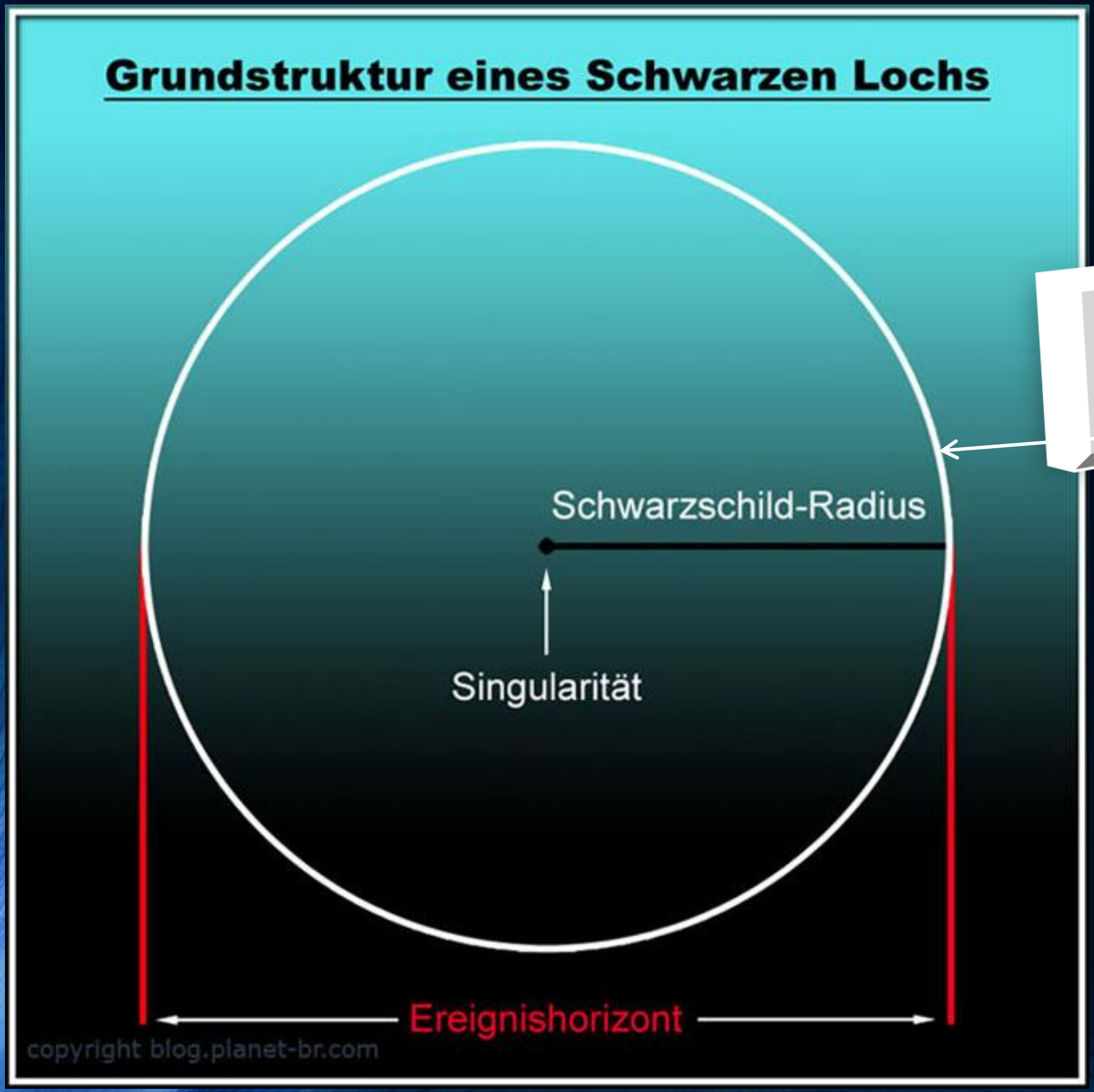


Abbildung 2.2: Raumzeitdiagramm der Schwarzschildmetrik in avancierten Eddington-Finkelstein Koordinaten.

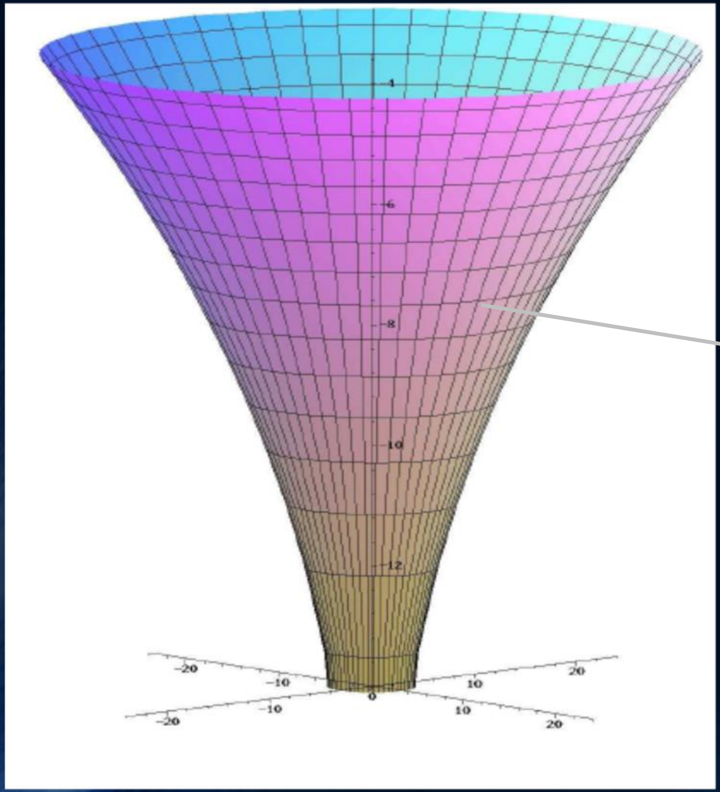
Der Ereignishorizont eines Schwarzen Loches



Das Bildnis des schwarzen Loches

(die wohl beste Veranschaulichung der wesentlichen Eigenschaften eines schwarzen Loches)

Der Raumzeit-Tricher im Reichstagsgebäude



Das Bildnis des schwarzen Loches

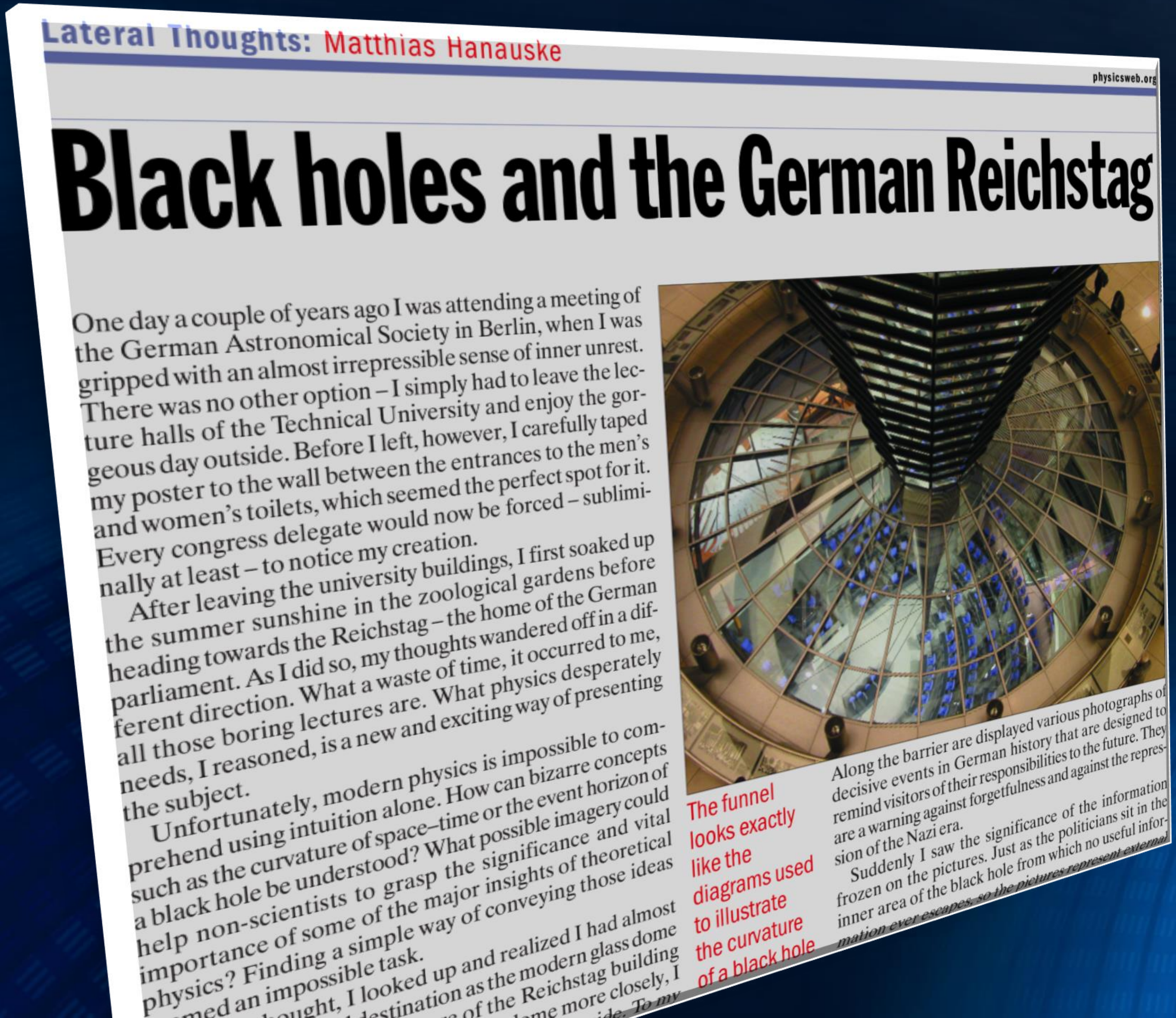
(die wohl beste Veranschaulichung der wesentlichen Eigenschaften eines schwarzen Loches)

Ereignis-
horizont

Das Bildnis des schwarzen Loches
(die wohl beste Veranschaulichung der wesentlichen Eigenschaften eines schwarzen Loches)

Der Aufzug im Reichstagstagsgebäude befindet sich ca. bei 3/2*R_S

Weiteres siehe: http://th.physik.uni-frankfurt.de/~hanauske/new/LateralThougts.html



Lateral Thoughts: Matthias Hanauske

physicsweb.org

Black holes and the German Reichstag

One day a couple of years ago I was attending a meeting of the German Astronomical Society in Berlin, when I was gripped with an almost irrepressible sense of inner unrest. There was no other option – I simply had to leave the lecture halls of the Technical University and enjoy the gorgeous day outside. Before I left, however, I carefully taped my poster to the wall between the entrances to the men's and women's toilets, which seemed the perfect spot for it. Every congress delegate would now be forced – subliminally at least – to notice my creation.

After leaving the university buildings, I first soaked up the summer sunshine in the zoological gardens before heading towards the Reichstag – the home of the German parliament. As I did so, my thoughts wandered off in a different direction. What a waste of time, it occurred to me, all those boring lectures are. What physics desperately needs, I reasoned, is a new and exciting way of presenting the subject.

Unfortunately, modern physics is impossible to comprehend using intuition alone. How can bizarre concepts such as the curvature of space–time or the event horizon of a black hole be understood? What possible imagery could help non-scientists to grasp the significance and vital importance of some of the major insights of theoretical physics? Finding a simple way of conveying those ideas seemed an impossible task.

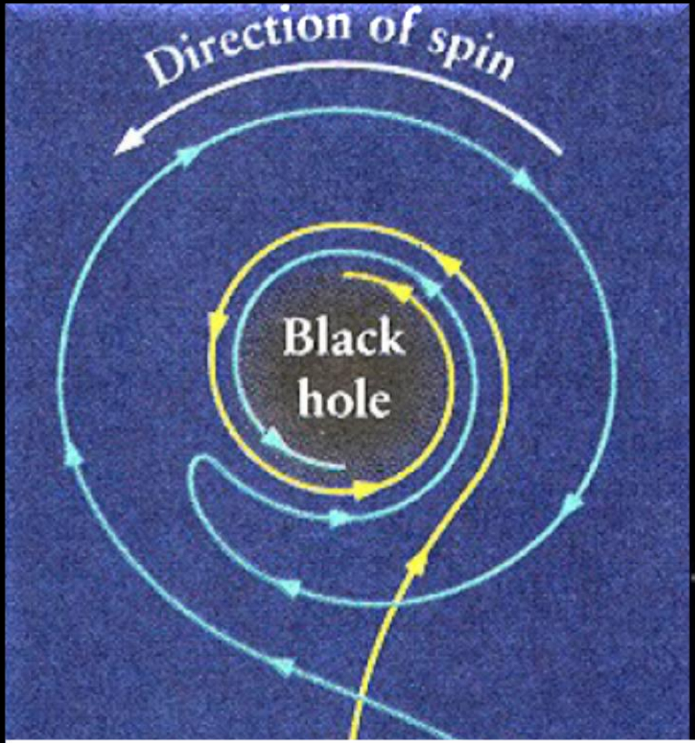
…thought, I looked up and realized I had almost … destination as the modern glass dome … of the Reichstag building … more closely, I … inside. To my

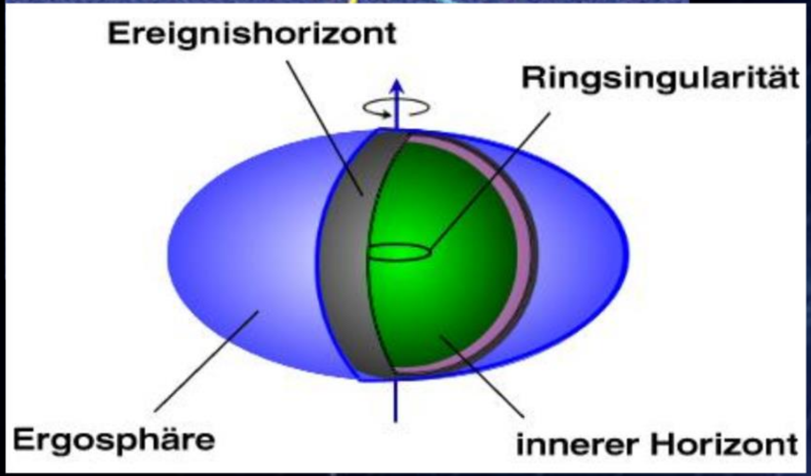
The funnel looks exactly like the diagrams used to illustrate the curvature of a black hole

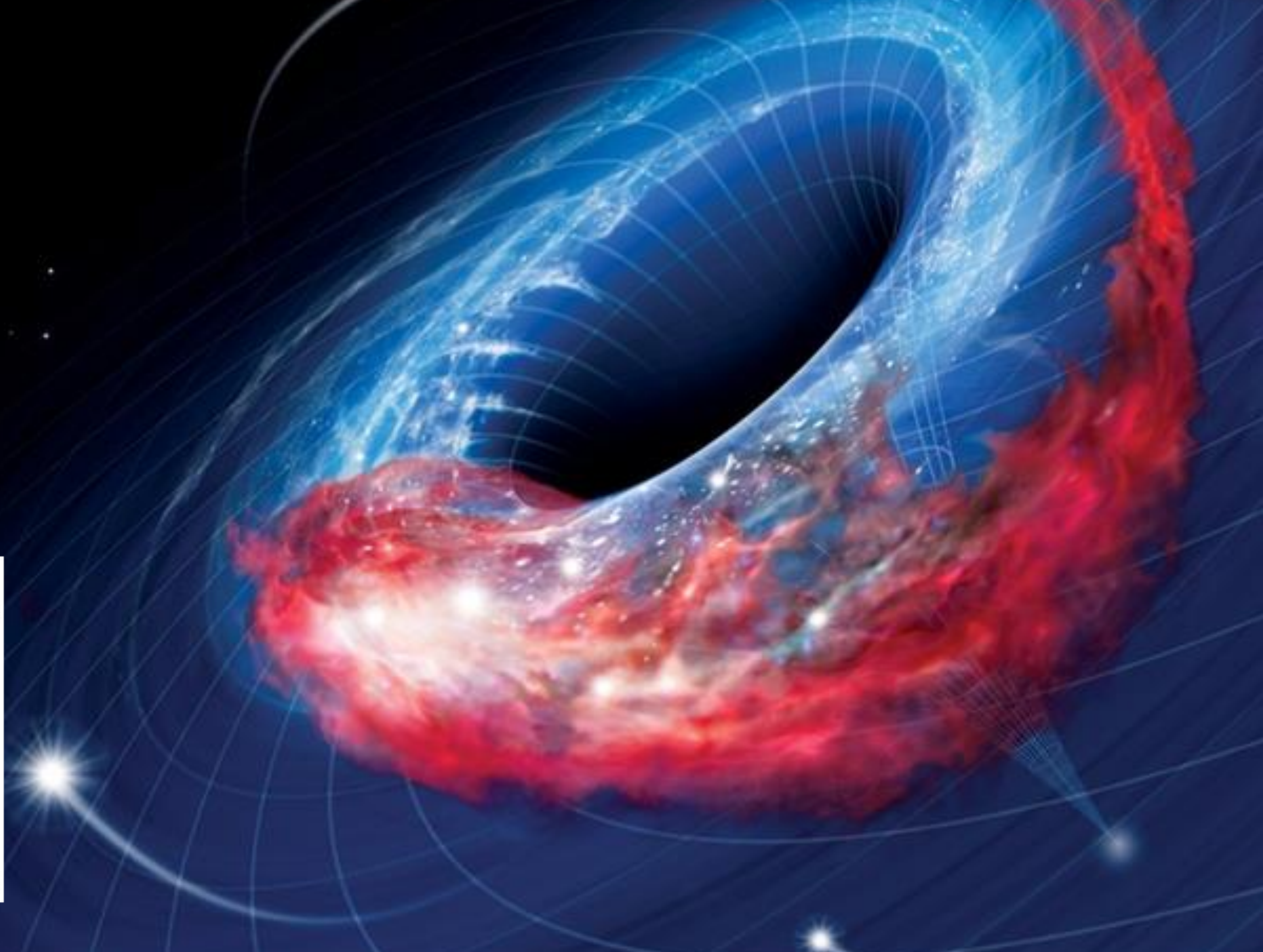
Along the barrier are displayed various photographs of decisive events in German history that are designed to remind visitors of their responsibilities to the future. They are a warning against forgetfulness and against the repression of the Nazi era.

Suddenly I saw the significance of the information frozen on the pictures. Just as the politicians sit in the inner area of the black hole from which no useful information ever escapes, so the pictures represent external

Rotierende schwarze Löcher







Wie sieht das schwarze Loch im Zentrum unserer Galaxie aus?

Das EU-Projekt **BlackHoleCam**

L.Rezzolla, H.Falke und M.Kramer

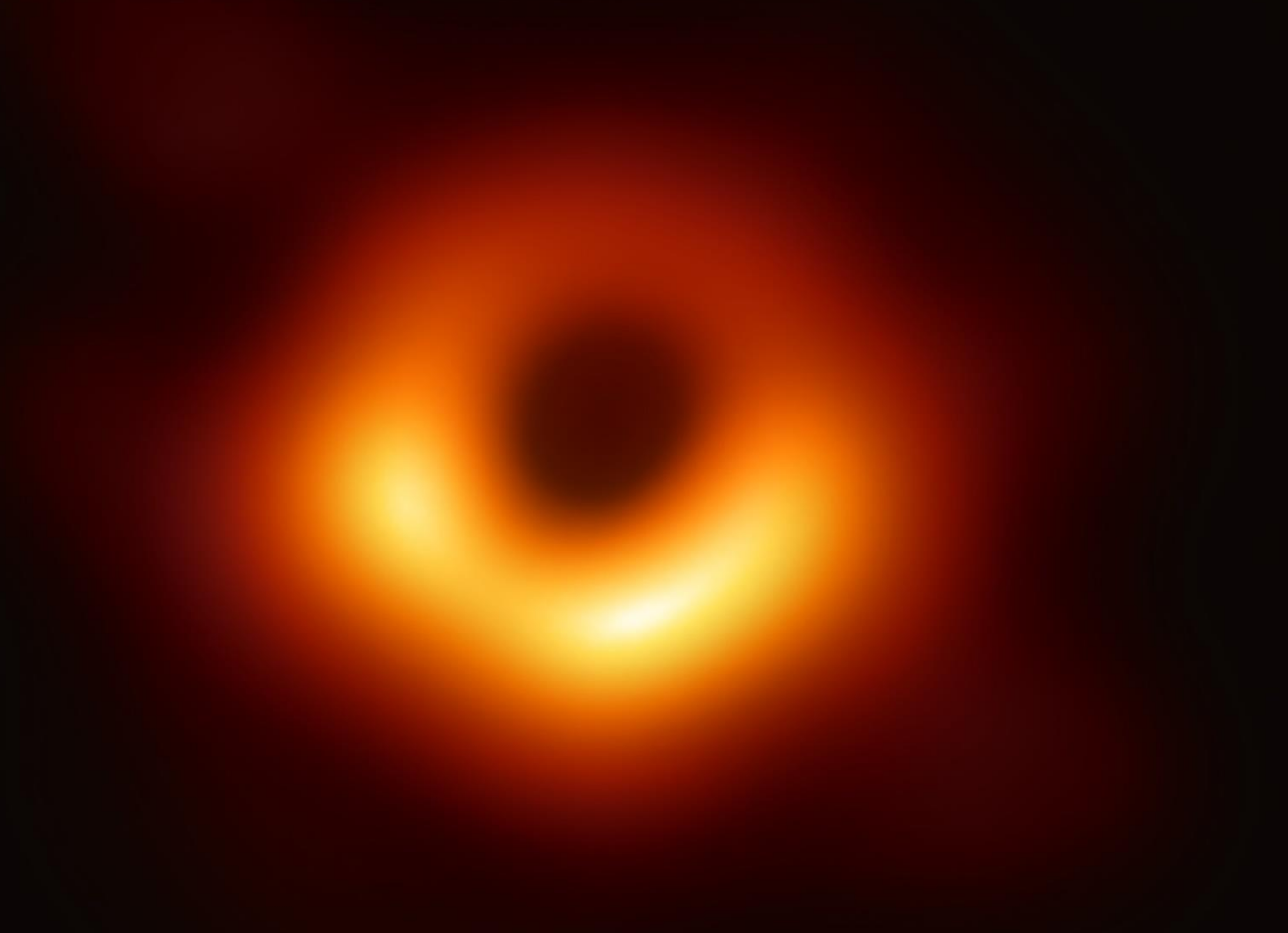
Black hole cam is a European funded project, which is a partner in the Event Horizon Telescope and not a separate network!

Das Bildnis des schwarzen Loches

(wie wird das wirkliche Bildnis des schwarzen Loches im Zentrum der Milchstrasse aussehen?)

Die ersten Bilder eines Schwarzen Lochs

Ein wenig mehr als hundert Jahre nachdem Albert Einstein seine Feldgleichungen der *Allgemeine Relativiätstheorie* der Öffentlichkeit präsentierte, und er damit die Grundlage für Gravitationswellen und schwarzer Löcher formulierte, ist seit einigen Wochen ein Meilenstein in der Geschichte der Astronomie in aller Munde (erstes Bild eines schwarzen Lochs, siehe rechte Abbildung).



YouTube Video: https://www.youtube.com/watch?v=Zh5p9Sr0_VU&list=PLn5gYfEKIag8nps1GKLqUW35AOgQY7aM2

Anlässlich der bahnbrechenden Aufnahme des ersten Bildes eines schwarzen Lochs im Zentrum unserer Nachbargalaxie M87, wurde am 17. April 2019 um 20 Uhr ein öffentlicher, populärwissenschaftlicher Abendvortrag im Otto Stern Zentrum (OSZ H1) am Campus Riedberg der Goethe Universität gehalten. Es sprachen die drei Principal Investigators des europäischen Black Hole Cam-Projekts (L.Rezzolla, M.Kramer und H.Falke).

The Einstein Equation and Neutron Stars

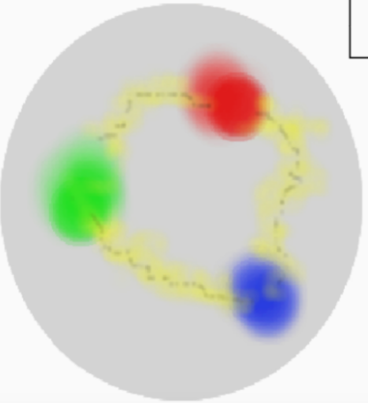
ART	Yang-Mills-Theories
$D_\beta v^\alpha = \partial_\beta\, v^\alpha + \Gamma^\alpha_{\sigma\beta}\, v^\sigma$	$D_{\beta a}{}^b = \partial_\beta\, 1_a{}^b + ig\, A_{\beta a}{}^b$
$R^\delta{}_{\mu\alpha\beta}\, v^\mu = [D_\alpha, D_\beta]\, v^\delta$ $R^\delta{}_{\mu\alpha\beta} = \Gamma^\delta_{\mu\alpha\vert\beta} - \Gamma^\delta_{\mu\beta\vert\alpha}$ $+\Gamma^\delta_{\nu\beta}\Gamma^\nu_{\mu\alpha} + \Gamma^\delta_{\nu\alpha}\Gamma^\nu_{\mu\beta}$	$F_{\alpha\beta a}{}^b = \frac{1}{ig}\left[D_{\alpha a}{}^c, D_{\beta c}{}^b\right]$ $= A_{\beta a}{}^b{}_{\vert\alpha} - A_{\alpha a}{}^b{}_{\vert\beta}$ $+\frac{1}{ig}\left[A_{\alpha a}{}^c, A_{\beta c}{}^b\right]$
$\mathcal{L}_G = R + \underbrace{\left(c_1\, R_{\mu\nu}R^{\mu\nu} + \ldots\right)}_{\equiv 0 \text{ for ART}}$	$\mathcal{L}_{YM} = \frac{1}{4} F_{\mu\nu a}{}^b\, F^{\mu\nu}{}_a{}^b$

QuantumCromoDynamic:

$(SU(3)_{(c)}$- Color Yang-Mills-Gauge Theory)

$D_{\beta A}{}^B = \partial_\beta\, 1_A{}^B + ig\, G_{\beta A}{}^B$

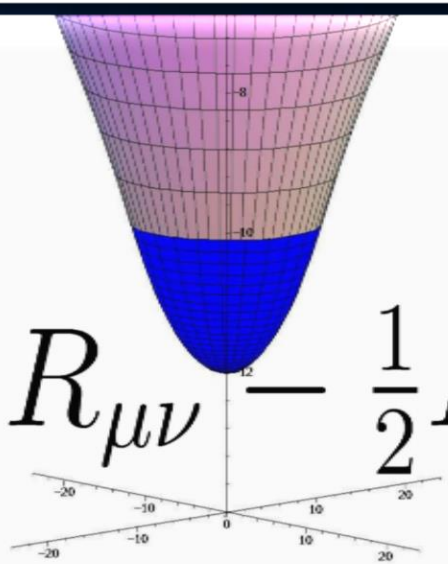
${}_A,{}^B$ =red, green, blue



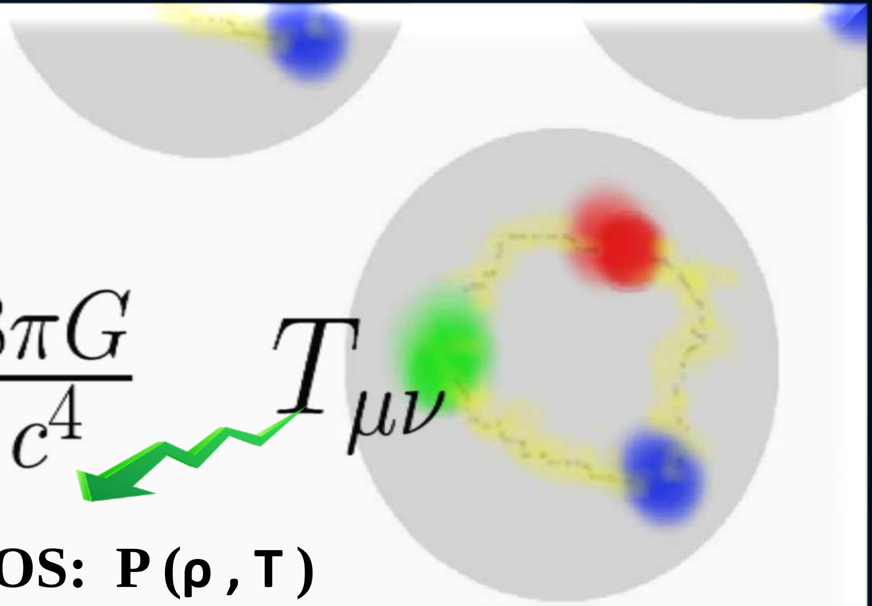
$$\psi^f_A = \begin{pmatrix} \psi^f_r \\ \psi^f_g \\ \psi^f_b \end{pmatrix}$$

Confinement

chiral symmetry, ...



$$R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu} = \frac{8\pi G}{c^4}\, T_{\mu\nu}$$



EOS: P (ρ , T)

Die Zustandsgleichung der Materie und das Quark-Gluon-Plasma

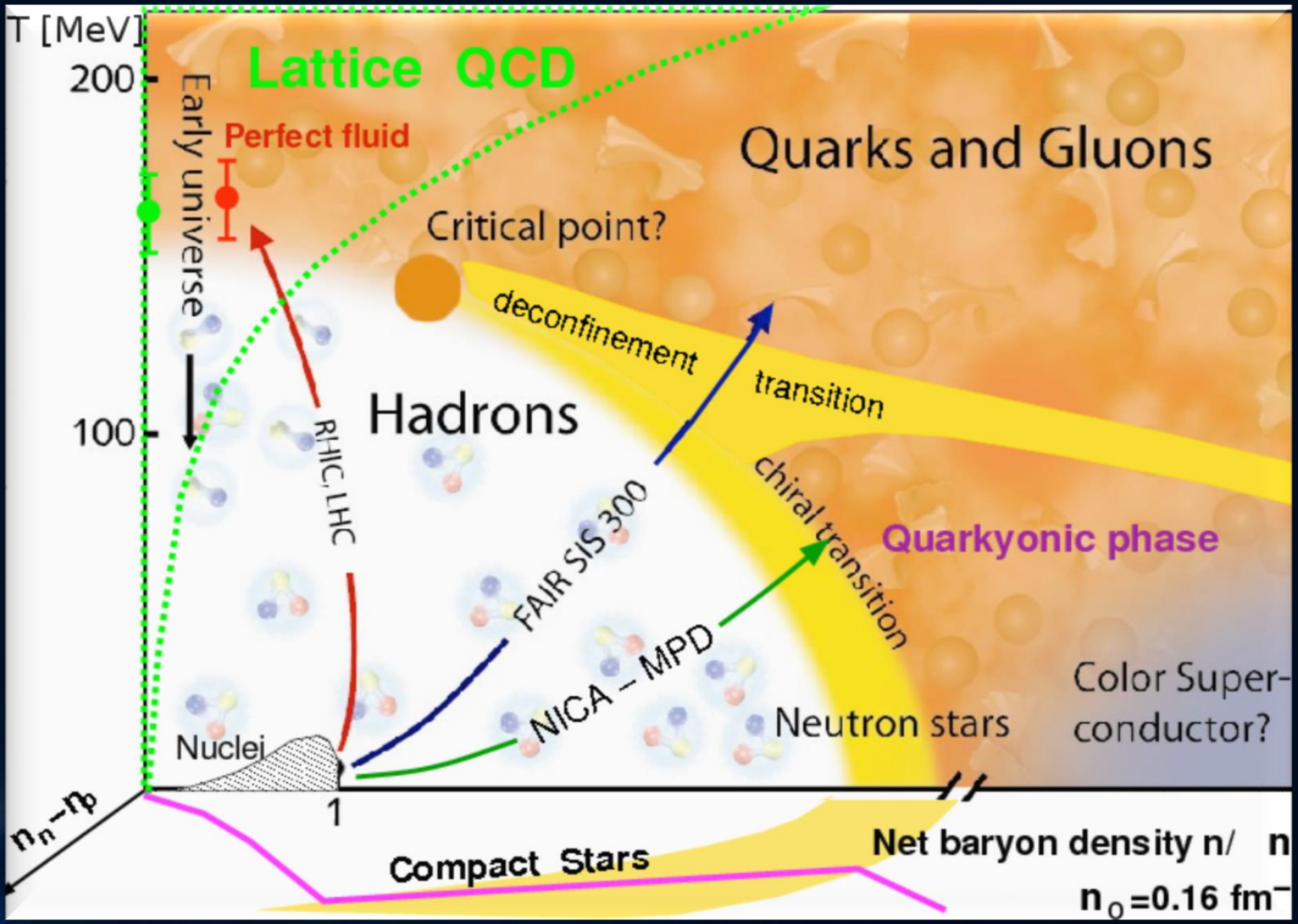
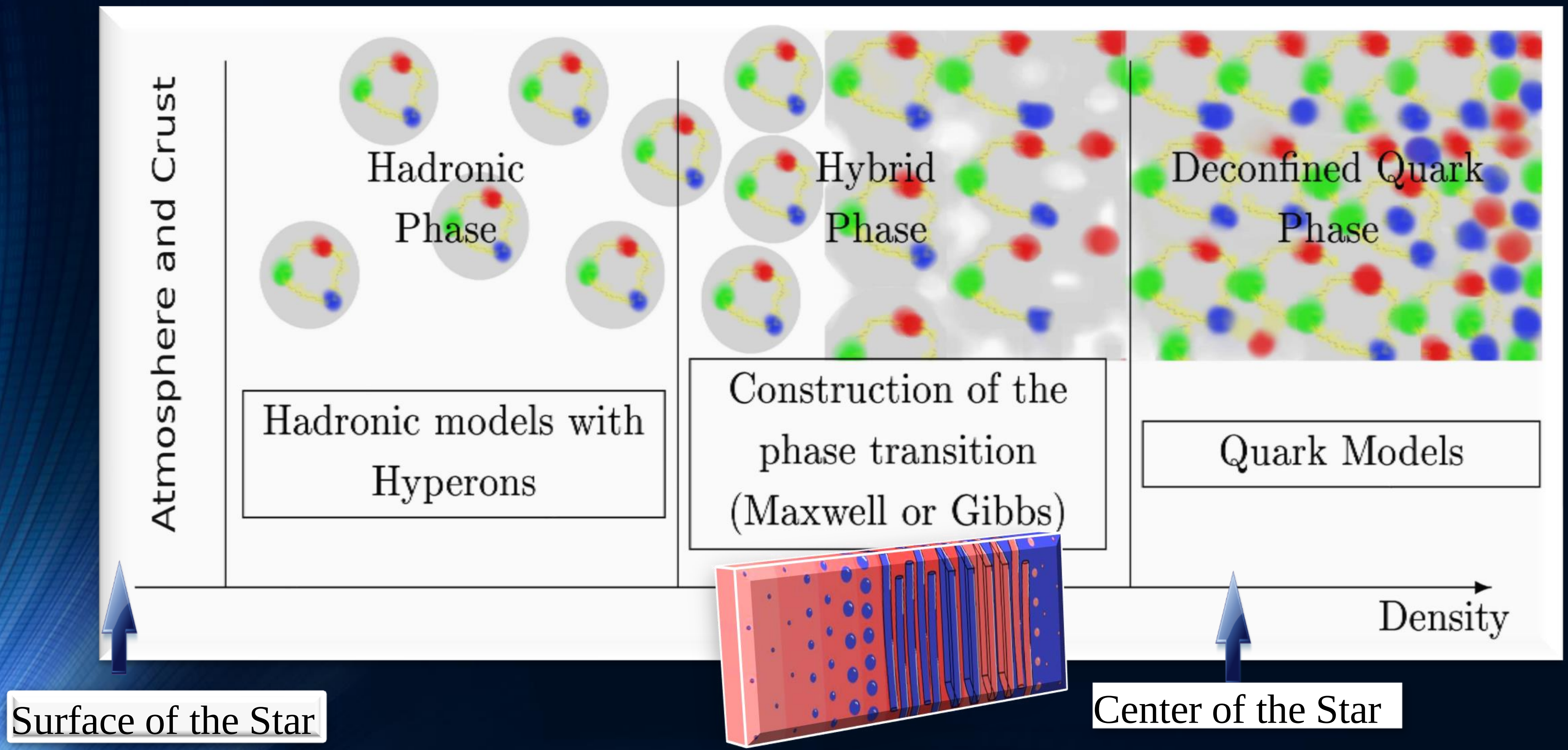


Image from http://inspirehep.net/record/823172/files/phd_qgp3D_quarkyonic2.png

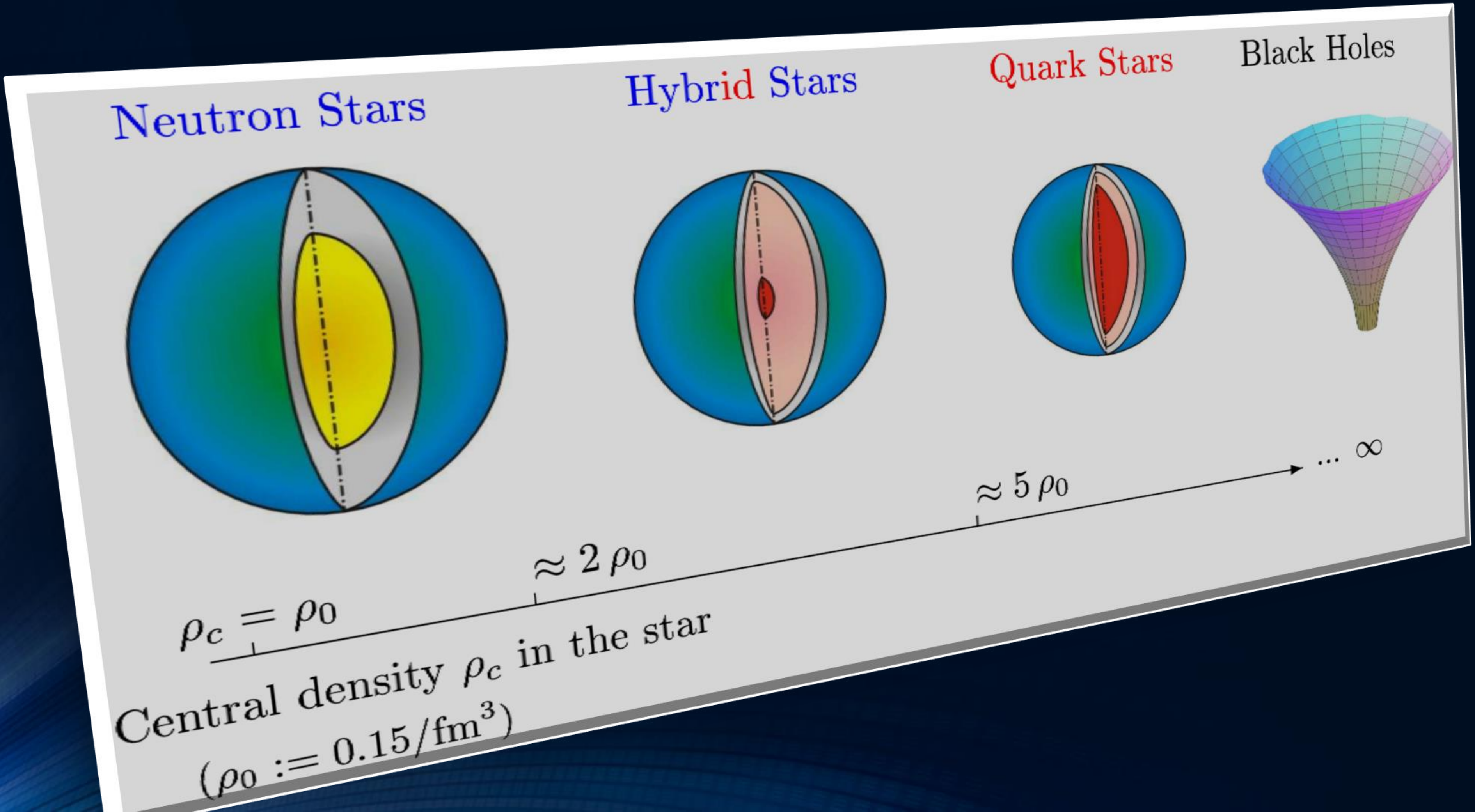
The QCD – Phase Transition and the Interior of a Hybrid Star



See: *Stable hybrid stars within a SU(3) Quark-Meson-Model*,
A.Zacchi, M.Hanauske, J.Schaffner-Bielich, PRD 93, 065011 (2016)

Neutronensterne, Quarksterne und schwarze Löcher

Bei welcher Dichte der Phasenübergang zum Quark-Gluon-Plasma einsetzt und welche Eigenschaften dieser Übergang im Detail hat ist weitgehend unbekannt. Theoretische Modellierung mittels unterschiedlicher effektiver Elementarteilchenmodelle.



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The equations of numerical relativity

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi T_{\mu\nu} \,, \qquad \text{(field equations)}$$

$$\nabla_\mu T^{\mu\nu} = 0 \,, \qquad \text{(cons. energy/momentum)}$$

$$\nabla_\mu (\rho u^\mu) = 0 \,, \qquad \text{(cons. rest mass)}$$

$$p = p(\rho, \epsilon, Y_e, \ldots) \,, \qquad \text{(equation of state)}$$

$$\nabla_\nu F^{\mu\nu} = I^\mu \,, \quad \nabla_\nu {}^{*}F^{\mu\nu} = 0 \,, \qquad \text{(Maxwell equations)}$$

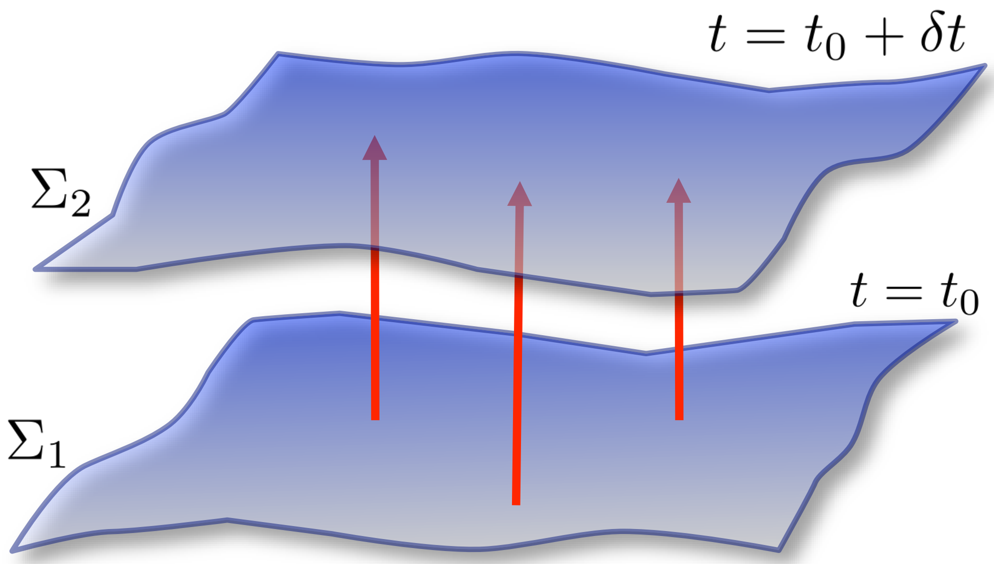
$$T_{\mu\nu} = T^{\rm fluid}_{\mu\nu} + T^{\rm EM}_{\mu\nu} + \ldots \qquad \text{(energy} - \text{momentum tensor)}$$

In GR these equations do not possess an analytic solution in the nonlinear regimes we are interested in

3+1splitting of spacetime

First step: foliate the 4D spacetime

Given a manifold $\mathcal{M}$ describing a spacetime with 4-metric $g_{\mu\nu}$ we want to foliate it via spacelike, three-dimensional hypersurfaces, i.e., $\Sigma_1, \Sigma_2, \ldots$ leveled by a scalar function. The time coordinate t is an obvious good choice.



Define therefore

$$\Omega_\mu \equiv \nabla_\mu t$$

such that

$$|\Omega|^2 \equiv g^{\mu\nu}\nabla_\mu t \nabla_\nu t = -\alpha^{-2}$$

This defines the "lapse" function which is strictly positive for spacelike hypersurfaces

$$\alpha(t, x^i) > 0$$

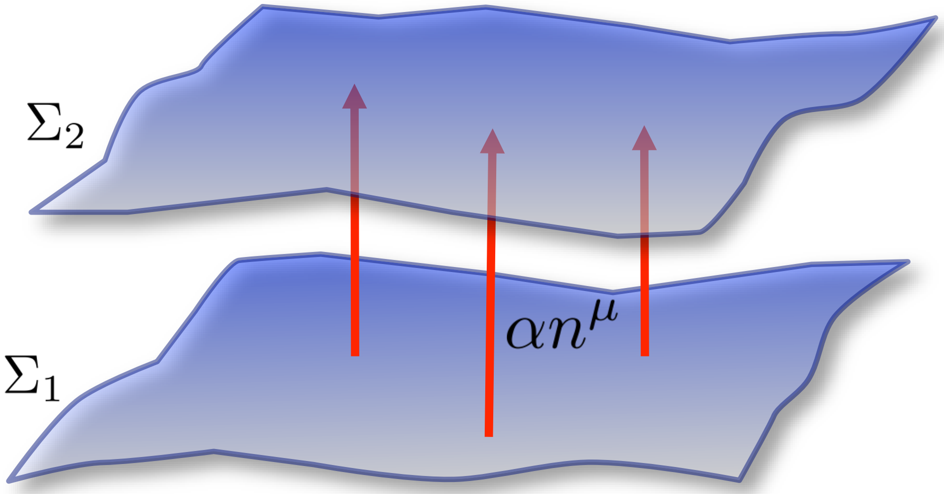
The lapse function allows then to do two important things:

i) define the unit normal vector to the hypersurface Σ

$$n^\mu \equiv -\alpha g^{\mu\nu}\Omega_\nu = -\alpha g^{\mu\nu}\nabla_\nu t$$

where

$$n^\mu n_\mu = -1$$



ii) define the spatial metric

$$\gamma_{\mu\nu} \equiv g_{\mu\nu} + n_\mu n_\nu$$

Second step: decompose 4-dim tensors

$\boldsymbol{n}$ and $\boldsymbol{\gamma}$ provide two useful tools to decompose any 4-dim. tensor into a purely spatial part (hence in Σ) and a purely timelike part (hence orthogonal to Σ and aligned with $\boldsymbol{n}$).

The spatial part is obtained after contracting with the spatial projection operator

$$\gamma^{\mu}{}_{\nu} = g^{\mu\alpha}\gamma_{\alpha\nu} = g^{\mu}{}_{\nu} + n^{\mu}n_{\nu} = \delta^{\mu}{}_{\nu} + n^{\mu}n_{\nu}$$

while the timelike part is obtained after contracting with the timelike projection operator

$$N^{\mu}{}_{\nu} = -n^{\mu}n_{\nu}$$

where the two projectors are obviously orthogonal

$$\gamma^{\nu}{}_{\mu}N^{\mu}{}_{\nu} = 0$$

It is now possible to define the 3-dim covariant derivative of a spatial tensor. This is simply the projection on Σ of all the indices of the the 4-dim. covariant derivative

$$D_\alpha T^\beta_{\ \delta} = \gamma^\rho_{\ \alpha}\gamma^\beta_{\ \sigma}\gamma^\tau_{\ \delta}\nabla_\rho T^\sigma_{\ \tau}$$

which, as expected, is compatible with spatial metric

$$D_\alpha \gamma^\beta_{\ \delta} = 0$$

All of the 4-dim tensor algebra can be extended straightforwardly to the 3-dim. spatial slice, so that the 3-dim covariant derivative can be expressed in terms of the 3-dimensional connection coefficients:

$$^{(3)}\Gamma^\alpha_{\ \beta\delta} = \frac{1}{2}\gamma^{\alpha\mu}(\gamma_{\mu\beta,\delta} + \gamma_{\mu\delta,\beta} - \gamma_{\beta\delta\,\mu})$$

Similarly, the 3-dim Riemann tensor associated with $\boldsymbol{\gamma}$ is defined via the double 3-dimensional covariant derivative of any spatial vector , ie $\boldsymbol{W}$

$$2D_{[\alpha}D_{\beta]}W_\delta = {}^{(3)}R^{\mu}{}_{\delta\alpha\beta}W_\mu$$

where

$${}^{(3)}R^{\mu}{}_{\delta\alpha\beta}n_\mu = 0 \qquad \text{and} \qquad 2T_{[\alpha\beta]} = T_{\alpha\beta} - T_{\beta\alpha}$$

More explicitly, the 3-dim Riemann tensor can be written in terms of the 3-dim connection coefficients as

$${}^{(3)}R^{\alpha}{}_{\beta\gamma\delta} = {}^{(3)}\Gamma^{\alpha}{}_{\beta\delta,\gamma} - {}^{(3)}\Gamma^{\alpha}{}_{\beta\gamma,\delta} + {}^{(3)}\Gamma^{\mu}{}_{\beta\delta}{}^{(3)}\Gamma^{\alpha}{}_{\mu\gamma} - {}^{(3)}\Gamma^{\mu}{}_{\beta\gamma}{}^{(3)}\Gamma^{\alpha}{}_{\mu\delta}$$

Also, the 3-dim contractions of the 3-dim Riemann tensor, i.e. the 3-dim Ricci tensor the 3-dim Ricci scalar are respectively given by

$${}^{(3)}R_{\alpha\beta} = {}^{(3)}R^{\delta}{}_{\alpha\delta\beta} \qquad\qquad {}^{(3)}R = {}^{(3)}R^{\delta}_{\delta}$$

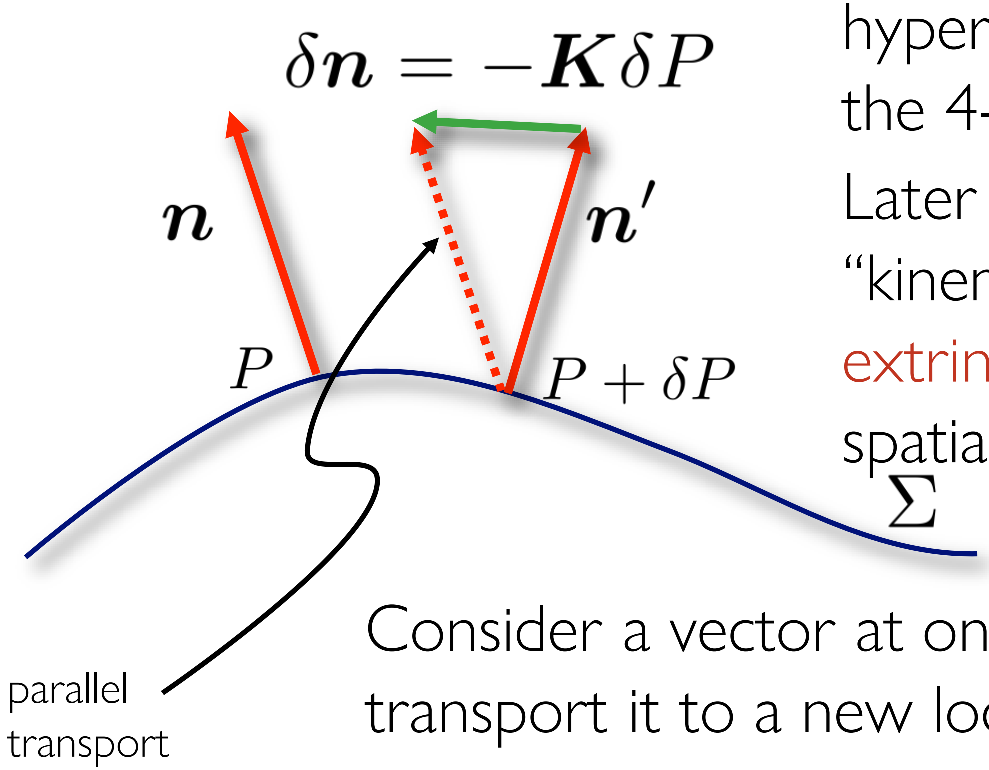
It is important not to confuse the 3-dim Riemann tensor ${}^{(3)}R^{\mu}{}_{\delta\alpha\beta}$ with the corresponding 4-dim one $R^{\mu}{}_{\delta\alpha\beta}$

${}^{(3)}R^{\mu}{}_{\delta\alpha\beta}$ is a 4-dimensional tensor but it is purely spatial (spatial derivatives of spatial metric γ)

$R^{\mu}{}_{\delta\alpha\beta}$ is a full 4-dimensional tensor containing also time derivatives of the full 4-dim metric $\boldsymbol{g}$

The information present in $R^{\mu}{}_{\delta\alpha\beta}$ and "missing" in ${}^{(3)}R^{\mu}{}_{\delta\alpha\beta}$ can be found in another spatial tensor: the extrinsic curvature.
As we shall see, this information is indeed describing the time evolution of the spatial metric

More geometrically, the extrinsic curvature measures the changes in the normal vector under parallel transport



Hence it measures how the 3-dim hypersurface is "bent" with respect to the 4-dim spacetime

Later on we will discuss also a "kinematical" interpretation of the extrinsic curvature in terms of the spatial metric

Consider a vector at one position P and parallel-transport it to a new location $P + \delta P$

The difference in the two vectors is proportional to the extrinsic curvature and this can be positive or negative

$$K_{\mu\nu} := -\gamma^{\lambda}{}_{\mu} D_{\lambda} n_{\nu}$$

Since the extrinsic curvature measures the bending of the spacelike hypersurface, two more equivalent definitions exists for the extrinsic curvature:

2) in terms of the acceleration of normal observers:

$$K_{\mu\nu} := -D_\mu n_\nu - n_\mu a_\nu = -D_\mu n_\nu - n_\mu n^\lambda D_\lambda n_\nu$$

3) in terms of the Lie derivative of the spatial metric:

$$K_{\mu\nu} := -\frac{1}{2}\mathcal{L}_{\boldsymbol{n}}\gamma_{\mu\nu}$$

1) in terms of the Lie derivative of the spatial metric:

$$K_{\mu\nu} := -\gamma^\lambda{}_\mu D_\lambda n_\nu$$

Finding a direction for evolutions

Note that the unit normal $\boldsymbol{n}$ to a spacelike hypersurface Σ is not the natural time derivative. This is because $\boldsymbol{n}$ is not dual to the surface 1-norm $\boldsymbol{\Omega}$, i.e.

$$n^\mu \Omega_\mu = n^\mu \nabla_\mu t = -\alpha \Omega^\mu \Omega_\mu = \frac{1}{\alpha}$$

We need therefore to find a new vector along which to carry out the time evolutions and that is dual to the surface 1-norm.
Such a vector is easily defined as

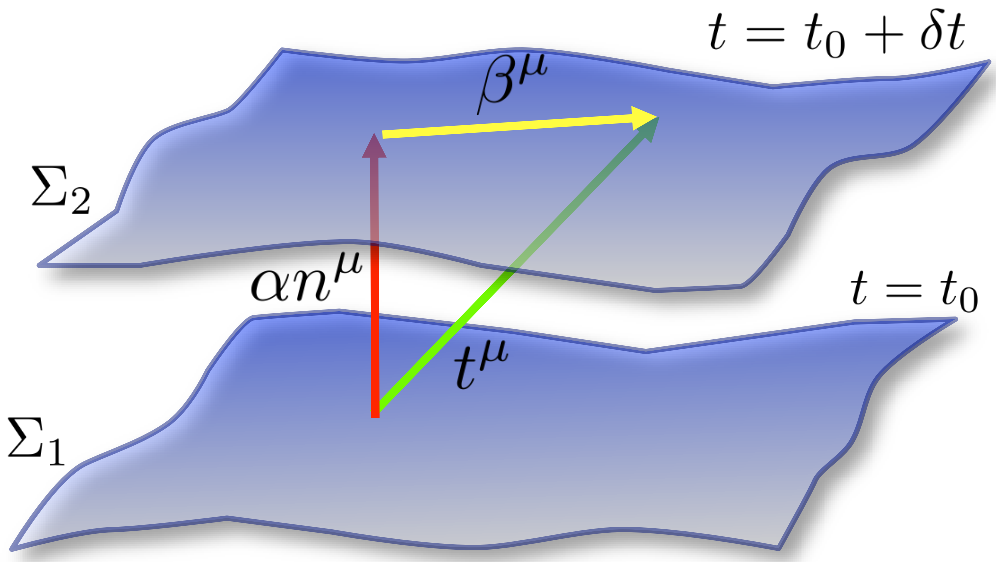
$$t^\mu \equiv \alpha n^\mu + \beta^\mu$$

where $\boldsymbol{\beta}$ is any spatial "shift" vector.

Clearly now the two tensors are dual to each other, ie

$$t^\mu \Omega_\mu = \alpha n^\mu \Omega_\mu + \beta^\mu \Omega_\mu = \alpha / \alpha = 1$$

Because the vector t^μ is dual to the 1-form Ω_μ, we are guaranteed that the integral curves of t^μ are naturally parametrized by the time coordinate.



Stated differently, all infinitesimal vectors t^μ originating on one hypersurface Σ_1 would end up on the same hypersurface Σ_2

This is not guaranteed for translations along Ω^μ

A more intuitive description of the lapse function α and of the shift vector β^μ will be presented once we introduce a coordinate basis

Note that t^μ is not necessarily timelike if the shift is superluminal

$$t^\mu t_\mu = -\alpha^2 + \beta^\mu \beta_\mu \lesseqgtr 0$$

Selecting a coordinate basis

So far we have dealt with tensor eqs and not specified a coordinate basis with unit vectors $\boldsymbol{e}_j$. Doing so can be useful to simplify equations and to highlight the "spatial" nature of $\boldsymbol{\gamma}$ and $\boldsymbol{K}$

The choice in this case in very simple. We want:
i) three of them have to be purely spatial, i.e.

$$n_\mu (\boldsymbol{e}_j)^\mu = 0 \qquad e.g. \qquad (\boldsymbol{e}_1)^\mu = (0, 1, 0, 0)$$

ii) the fourth one has to be along the vector $\boldsymbol{t}$, i.e.

$$(\boldsymbol{e}_0)^\mu = t^\mu = (1, 0, 0, 0)$$

As a result:

$$\mathcal{L}_{\boldsymbol{t}} = \partial_t$$

i.e. the Lie derivative along $\boldsymbol{t}$ is a simple partial derivative

$$n_j = n_\mu (e_j)^\mu = 0 \quad \text{but} \quad n_0 \neq 0$$

i.e. the space covariant components of a **timelike** vector are zero; only the covariant time component survives

$$n_\mu \beta^\mu = \beta^0 n_0 = 0 \quad \Longrightarrow \quad \beta^0 = 0 \quad \Longrightarrow \quad \beta^\mu = (0, \beta^j)$$

i.e. the time contravariant component of a **spacelike** vector is zero; only the spatial contravariant components survive

Putting things together and bearing in mind that $n_\mu n^\mu = -1$

$$n^\mu = \frac{1}{\alpha}(1, -\beta^i)\,; \qquad n_\mu = (-\alpha, 0, 0, 0)$$

Because for any spatial tensor $T^{\mu 0} = 0$ the contravariant components of the metric in a 3+1 split are

$$g^{\mu\nu} = \begin{pmatrix} -1/\alpha^2 & \beta^i/\alpha^2 \\ \beta^i/\alpha^2 & \gamma^{ij} - \beta^i\beta^j/\alpha^2 \end{pmatrix}$$

Similarly, since $g_{ij} = \gamma_{ij}$ the covariant components are

$$g_{\mu\nu} = \begin{pmatrix} -\alpha^2 + \beta_i\beta^i & \beta_i \\ \beta_i & \gamma_{ij} \end{pmatrix}$$

Note that $\gamma^{ik}\gamma_{kj} = \delta^i{}_j$ (i.e. γ^{ij}, γ_{ij} are inverses) and thus they can be used to raise/lower the indices of **spatial** tensors

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Decomposing the Einstein equations

- So far we have just played with differential geometry. No mention has been made of the Einstein equations.

- The 3+1 splitting naturally "splits" the Einstein equations into:
  - a set which is fully defined on each spatial hypersurfaces (and does not involve therefore time derivatives).
  - a set which instead relates quantities (i.e. spatial metric and extrinsic curvature) between two adjacent hypersurfaces.

- The first set is usually referred to as the "constraint" equations, while the second one as the "evolution" equation

Next, we need to decompose the Einstein equations in the spatial and timelike parts.

$$G_{\mu\nu} := R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi T_{\mu\nu}$$

and to do this we need to define a few identities

First we decompose the 4-dim Riemann tensor $R_{\alpha\beta\mu\nu}$ projecting all indices to obtain the Gauss equations

$$^{(3)}R_{\alpha\beta\gamma\delta} + K_{\alpha\gamma}K_{\beta\delta} - K_{\alpha\delta}K_{\beta\gamma} = \gamma^{\mu}{}_{\alpha}\gamma^{\nu}{}_{\beta}\gamma^{\rho}{}_{\delta}\gamma^{\sigma}{}_{\gamma}\, R_{\mu\nu\sigma\rho}$$

Next, we make 3 spatial projections and a timelike one to obtain the Codazzi equations

$$D_{\alpha}K_{\beta\gamma} - D_{\beta}K_{\alpha\gamma} = \gamma^{\rho}{}_{\beta}\gamma^{\mu}{}_{\alpha}\gamma^{\nu}{}_{\gamma}n^{\sigma}R_{\mu\nu\sigma\rho}$$

Finally we take 2 spatial projections and 2 timelike ones to obtain the Ricci equations

$$\mathcal{L}_{\boldsymbol{n}} K_{\alpha\beta} = n^\delta n^\gamma \gamma^\mu{}_\alpha \gamma^\nu{}_\beta R_{\nu\delta\mu\gamma} - \frac{1}{\alpha} D_\alpha D_\beta \alpha - K^\gamma{}_\beta K_{\alpha\gamma}$$

where the second derivative of the lapse has been introduced via the identity

$$a_\mu = D_\mu \ln \alpha$$

Another important identity which will be used in the following is

$$D_\mu U^\nu = \gamma_\mu{}^\rho \nabla_\rho U^\nu + K_{\mu\rho} U^\rho n^\nu$$

and which holds for any spatial vector $\boldsymbol{U}$ $(U^\mu n_\mu = 0)$

The evolution part of the Einstein equations

We are now ready to express the missing piece of the 3+1 decomposition and derive the evolution part of the Einstein eqs.

We need suitable projections of the right-hand-side of the Einstein equations and in particular the two spatial ones, ie

$$\gamma^{\mu}{}_{\alpha}\gamma^{\nu}{}_{\beta}G_{\mu\nu} = 8\pi S_{\alpha\beta} \equiv 8\pi\gamma^{\mu}{}_{\alpha}\gamma^{\nu}{}_{\beta}T_{\mu\nu}$$

where the energy-momentum tensor of a perfect fluid is:

$$T_{\mu\nu} = (e+p)u_{\mu}u_{\nu} + pg_{\mu\nu} = h\rho u_{\mu}u_{\nu} + pg_{\mu\nu}$$

with

ρ : rest-mass density

p : pressure

ϵ : specific internal energy

$e = \rho(1+\epsilon)$: total energy density

$h = \dfrac{e+p}{\rho}$: specific enthalpy

$S \equiv S^{\mu}{}_{\mu}$

Since $n^\mu u_\mu = 1$, (the two vectors are parallel and unit vectors) the energy density measured by the normal observers will be given by the double timelike projection

$$e = n^\mu n^\nu T_{\mu\nu}$$

Similarly, the momentum density (i.e. the extension of the Newtonian mass current) will be given by the mixed time and spatial projection

$$j_\mu = -\gamma^\alpha{}_\mu n^\beta T_{\alpha\beta} = -(h\rho u_\mu + p n_\mu)$$

Just as a reminder, the fully spatial projection of the energy-momentum tensor was already introduced as

$$S_{\mu\nu} = \gamma^\alpha{}_\mu \gamma^\beta{}_\nu T_{\alpha\beta}$$

The (ADM) Einstein eqs in 3+1

In such a foliation, we can write the Einstein eqs in the 3+1 splitting of spacetime in a set of evolution and constraint equations as:

$$\gamma \cdot \gamma \cdot (\text{Einstein eqs}) \;+\; \text{Ricci eqs} \Longrightarrow$$

$$\begin{aligned}\partial_t K_{ij} = & -D_i D_j \alpha + \alpha(R_{ij} - 2K_{ik}K^{kj} + KK_{ij}) \\ & -8\pi\alpha(R_{ij} - \frac{1}{2}\gamma_{ij}(S-e)) + \mathcal{L}_{\boldsymbol{\beta}} K_{ij}\end{aligned}$$

[6 eqs]

$$\partial_t \gamma_{ij} = -2\alpha K_{ij} + \mathcal{L}_{\boldsymbol{\beta}} \gamma_{ij}$$

[6 eqs]

These are 12 hyperbolic, first-order in time, second-order in space, nonlinear partial differential equations: evolution equations

The constraint equations (I)

We first time-project twice the left-hand-side of the Einstein equations to obtain

$$2n^\mu n^\nu G_{\mu\nu} = {}^{(3)}R + K^2 - K_{\mu\nu}K^{\mu\nu}$$

Doing the same for the right-hand-side, using the Gauss eqs contracted twice with the spatial metric and the definition of the energy density we finally reach the form of the Hamiltonian constraint equation

$$ {}^{(3)}R + K^2 - K_{\mu\nu}K^{\mu\nu} = 16\pi e$$

Note that this is a single elliptic equation (hence not containing time derivative) which should be satisfied everywhere on the spatial hypersurface Σ

The constraint equations (II)

Similarly, with a mixed time-space projection of the left-hand-side of the Einstein equations we obtain

$$-\gamma^{\mu}{}_{\alpha} n^{\nu} G_{\mu\nu} = -\overset{(3)}{R}_{\alpha\nu} n^{\nu} + \frac{1}{2} n_{\alpha} R$$

Doing the same for the right-hand-side, using the contracted Codazzi equations and the definition of the momentum density we finally reach the form of the momentum constraint equations

$$D_{\nu} K^{\nu}{}_{\mu} - D_{\mu} K = 8\pi j_{\mu}$$

which are also 3 elliptic equations.

The 4 constraint equations are the necessary and sufficient integrability conditions for the embedding of the spacelike hypersurfaces $(\Sigma, \gamma_{\mu\nu}, K_{\mu\nu})$ in the 4-dim. spacetime $(\mathcal{M}, g_{\mu\nu})$

The (ADM) Einstein eqs in 3+1

Similarly

$$\boldsymbol{n} \cdot \boldsymbol{n} \cdot (\text{Einstein eqs}) \; + \; \text{Gauss eqs} \Longrightarrow$$

$$R + K^2 - K_{ij}K^{ij} = 16\pi e$$

Hamiltonian Constraint (HC) [1 eq]

$$\boldsymbol{\gamma} \cdot \boldsymbol{n} \cdot (\text{Einstein eqs}) \; + \; \text{Codazzi eqs} \Longrightarrow$$

$$D_j K^j{}_i - D_i K = 8\pi j_i$$

Momentum Constraints (MC) [3 eqs]

These are 1+3 elliptic (second-order in space), nonlinear partial differential equations: constraint equations

The (ADM) Einstein eqs in 3+1

All together we have:

$$\partial_t K_{ij} = -D_i D_j \alpha + \alpha(R_{ij} - 2K_{ik}K^{kj} + KK_{ij}) \\ -8\pi\alpha(R_{ij} - \frac{1}{2}\gamma_{ij}(S - e)) + \mathcal{L}_{\boldsymbol{\beta}} K_{ij}$$

[6]

$$\partial_t \gamma_{ij} = -2\alpha K_{ij} + \mathcal{L}_{\boldsymbol{\beta}} \gamma_{ij}$$

[6]

$$R + K^2 - K_{ij}K^{ij} = 16\pi e$$

[1]

$$D_j K^j{}_i - D_i K = 8\pi j_i$$

[3]

These 6+6 (+3+1) eqs are also known as the ADM equations. In practice, only the evolution eqs are solved and the constraints are instead monitored (more later)

ADM vs Maxwell

The ADM eqs may appear as rather cryptic and simply complicated. However, it is easy to see analogies with the Maxwell eqs. and make the equations less cryptic.

The relevant quantities in this case are the electric and magnetic fields $\boldsymbol{E}, \boldsymbol{B}$, the charge density ρ_e and the charge current density $\boldsymbol{J}$

Then also the Maxwell equations split into evolution equations

$$\partial_t \boldsymbol{E} = \nabla \times \boldsymbol{B} - 4\pi \boldsymbol{J} \,, \quad \Longleftrightarrow \quad \partial_t E_i = \epsilon_{ijk} D^j B^k - 4\pi J_i \,,$$

$$\partial_t \boldsymbol{B} = -\nabla \times \boldsymbol{E} \,, \quad \Longleftrightarrow \quad \partial_t B_i = -\epsilon_{ijk} D^j E^k$$

and constraint equations

$$\nabla \cdot \boldsymbol{E} = 4\pi \rho_e \,, \quad \Longleftrightarrow \quad \partial_i E^i = 4\pi \rho_e \,,$$

$$\nabla \cdot \boldsymbol{B} = 0 \,, \quad \Longleftrightarrow \quad \partial_i B^i = 0 \,,$$

It is then possible to make the associations

$$\begin{array}{ccc} \Phi & \longleftrightarrow & \beta_i \\ A_i & \longleftrightarrow & \gamma_{ij} \\ E_i & \longleftrightarrow & K_{ij} \end{array}$$

and realize that the RHSs of the evolution equation of A_i/γ_{ij} involve a field variable E_i/K_{ij} and the spatial derivatives of a gauge quantity Φ/β_i

Similarly, the RHS of the evolution equation of E_i/K_{ij} involve matter sources as well as second spatial derivatives of the second field variable A_i/γ_{ij}

Indeed, the similarities between the ADM eqs and the Maxwell eqs written in terms of the vector potential (i.e. as in previous slide) are so large that they suffer of the same problems/instabilities (more later)

In practice, the ADM are essentially never used!

These equations are perfectly alright mathematically but not in a form that is well suited for numerical implementation.

Indeed the system can be shown to be weakly hyperbolic and hence "ill-posed"

In practice, numerical instabilities rapidly appear that destroy the solution exponentially

However, the stability properties of numerical implementations can be improved by introducing certain new auxiliary functions and rewriting the ADM equations in terms of these functions.

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The same is done for the ADM eqs and new evolution variables are introduced to obtain a set of eqs that is strongly hyperbolic and hence well-posed (doesn't blow up).

$$\phi = \frac{1}{12}\ln(\det(\gamma_{ij})) = \frac{1}{12}\ln(\gamma),$$
$$\tilde{\gamma}_{ij} = e^{-4\phi}\gamma_{ij},$$
$$K = \gamma^{ij}K_{ij},$$
$$\tilde{A}_{ij} = e^{-4\phi}(K_{ij} - \frac{1}{3}\gamma_{ij}K),$$
$$\Gamma^i = \gamma^{jk}\Gamma^i_{jk}$$
$$\tilde{\Gamma}^i = \tilde{\gamma}^{jk}\tilde{\Gamma}^i_{jk}$$

ϕ: conformal factor

$\tilde{\gamma}_{ij}$: conformal 3-metric

K: trace of extrinsic curvature

$\tilde{A}_{ij}$: trace-free conformal extrinsic curvature

$\tilde{\Gamma}^i$:"Gammas"

are our new evolution variables

The ADM equations are then rewritten as

$$\mathcal{D}_t\tilde{\gamma}_{ij} = -2\alpha\tilde{A}_{ij} \ , \qquad \text{where } \mathcal{D}_t \equiv \partial_t - \mathcal{L}_\beta$$

$$\mathcal{D}_t\phi = -\frac{1}{6}\alpha K \ ,$$

$$\mathcal{D}_t\tilde{A}_{ij} = e^{-4\phi}\left[-\nabla_i\nabla_j\alpha + \alpha\left(R_{ij} - S_{ij}\right)\right]^{\mathrm{TF}} + \alpha\left(K\tilde{A}_{ij} - 2\tilde{A}_{il}\tilde{A}^l_j\right) \ ,$$

$$\mathcal{D}_t K = -\gamma^{ij}\nabla_i\nabla_j\alpha + \alpha\left[\tilde{A}_{ij}\tilde{A}^{ij} + \frac{1}{3}K^2 + \frac{1}{2}\left(\rho + S\right)\right] \ ,$$

$$\mathcal{D}_t\tilde{\Gamma}^i = -2\tilde{A}^{ij}\partial_j\alpha + 2\alpha\left(\tilde{\Gamma}^i_{jk}\tilde{A}^{kj} - \frac{2}{3}\tilde{\gamma}^{ij}\partial_j K - \tilde{\gamma}^{ij}S_j + 6\tilde{A}^{ij}\partial_j\phi\right)$$
$$-\partial_j\left(\beta^l\partial_l\tilde{\gamma}^{ij} - 2\tilde{\gamma}^{m(j}\partial_m\beta^{i)} + \frac{2}{3}\tilde{\gamma}^{ij}\partial_l\beta^l\right) \ .$$

These equations are also known as the BSSNOK equations or more simply the conformal traceless formulation of the Einstein equations.

Although not self evident, the BSSNOK equations are strongly hyperbolic with a structure which is resembling the 1st-order in time, 2nd-order in space formulation

$$\Box\phi = 0 \quad \iff \begin{cases} \partial_t \phi = \psi \\ \partial_t \psi = \partial^i \partial_i \phi \end{cases}$$

scalar wave equation

$$\begin{cases} \partial_t \tilde{\gamma}_{ij} \propto \tilde{A}_{ij} \\ \partial_t \tilde{A}_{ij} \propto D^i D_i \tilde{\gamma}_{ij} \end{cases}$$

BSSNOK

The BSSNOK is a widely used formulation of the Einstein eqs and used to simulate black holes and neutron stars. Other formulations have been recently suggested that have even better properties, e.g. CCZ4, Z4c.

Recap (I)

- ☑ The 3+1 splitting of the 4-dim spacetime represents an effective way to perform numerical solutions of the Einstein eqs.
- ☑ Such a splitting amounts to projecting all 4-dim. tensors either on spatial hypersurfaces or along directions orthogonal to such hypersurfaces.
- ☑ The 3-metric and the extrinsic curvature describe the properties of each slice.
- ☑ Two functions, the lapse and the shift, tell how to relate coordinates between two slices: the lapse measures the proper time, while the shift measures changes in the spatial coords.
- ☑ Einstein equations naturally split into evolution equations and constraint equations.

Recap (II)

- ☑ The ADM eqs are ill posed and not suitable for numerics.
- ☑ Alternative formulations (BSSNOK, CCZ4, Z4c) have been developed that are strongly hyperbolic and hence well-posed.
- ☑ Both formulations make use of the constraint equations and can use additional evolution equations to damp the violationsns
- ☑ The hyperbolic evolution eqs. to solve are: 6+6+(3+1+1) = 17. We also "compute" 3+1=4 elliptic constraint eqs

$$\mathcal{H} \equiv {}^{(3)}R + K^2 - K_{ij}K^{ij} = 0 \ , \quad \text{(Hamiltonian constraint)}$$

$$\mathcal{M}^i \equiv D_j(K^{ij} - g^{ij}K) = 0 \ , \quad \text{(momentum constraints)}$$

NOTE: these eqs are not solved but only monitored to verify

$$||\mathcal{H}|| \simeq ||\mathcal{M}^i|| < \varepsilon \sim 10^{-4} - 10^{-2}$$

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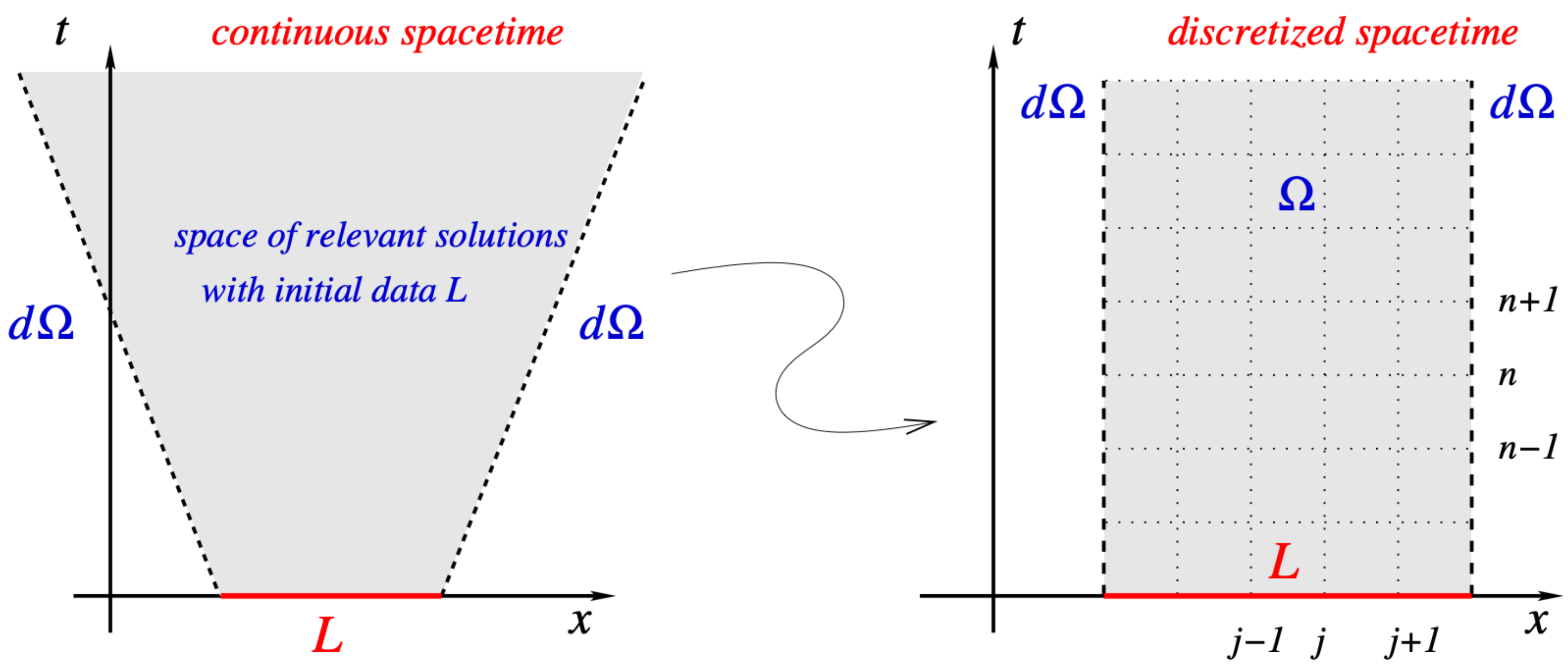
Initial data

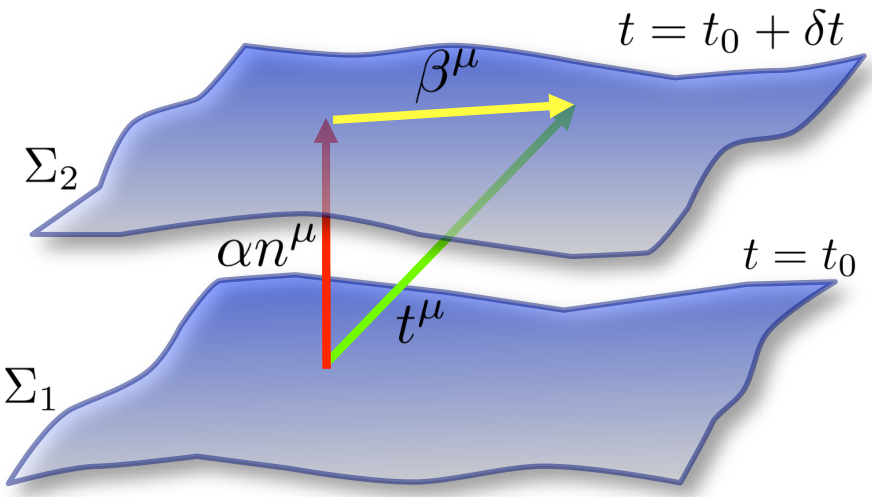
Einstein equations represent an initial-value boundary problem (IVBP). Stated differently, once the solution is known/specified at any initial time, the hyperbolic nature of the equations completely determines the space of future solutions



Gauge conditions

Let us recap what we have already seen for the interpretation ofthe lapse, shift and spatial metric. Using the expression for the covariant 4-dim covariant metric, the line element is given

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\mu = -(\alpha^2 - \beta^i\beta_i)\mathrm{d}t^2 + 2\beta_i \mathrm{d}x^i \mathrm{d}t + \gamma_{ij}\mathrm{d}x^i \mathrm{d}x^j$$



Hence:

the lapse measures proper time between two adjacent hypersurfaces

$$\mathrm{d}\tau^2 = -\alpha^2(t, x^j)\mathrm{d}t^2$$

the shift relates spatial coordinates between two adjacent hypersurfaces

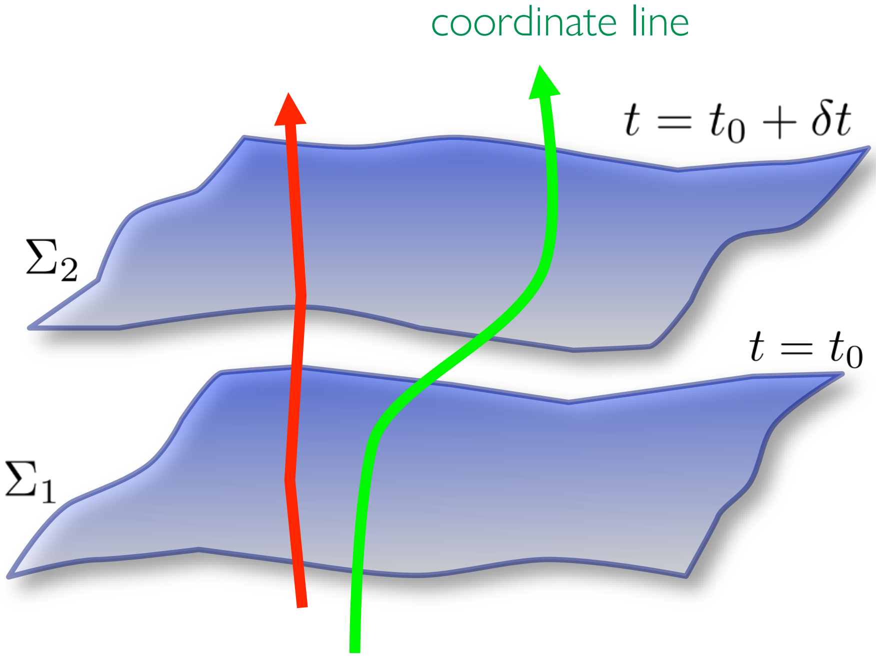
$$x^i_{t_0+\delta t} = x^i_{t_0} - \beta^i(t, x^j)\mathrm{d}t$$

the spatial metric measures distances between points on every hypersurface

$$\mathrm{d}l^2 = \gamma_{ij}\mathrm{d}x^i \mathrm{d}x^j$$

We can now have a more intuitive interpretation of the lapse, shift and spatial metric. Using the expression for the covariant 4-dim covariant metric, the line element is given

$$ds^2 = g_{\mu\nu}dx^\mu dx^\mu = -(\alpha^2 - \beta^i\beta_i)dt^2 + 2\beta_i dx^i dt + \gamma_{ij}dx^i dx^j$$



Hence:

the lapse measures proper time between two adjacent hypersurfaces

$$d\tau^2 = -\alpha^2(t, x^j)dt^2$$

the shift relates spatial coordinates between two adjacent hypersurfaces

$$x^i_{t_0+\delta t} = x^i_{t_0} - \beta^i(t, x^j)dt$$

the spatial metric measures distances between points on every hypersurface

$$dl^2 = \gamma_{ij}dx^i dx^j$$

NOTE: the lapse, and shift are not solutions of the Einstein equations but represent our “gauge freedom”, namely the freedom (arbitrariness) in which we choose to foliate the spacetime.

Any prescribed choice for the lapse is usually referred to as a ”slicing condition”, while any choice for the shift is usually referred to as ”spatial gauge condition”

While there are infinite possible choices, not all of them are equally useful to carry out numerical simulations. Indeed, there is a whole branch of numerical relativity that is dedicated to finding suitable gauge conditions.

Several possible routes are possible

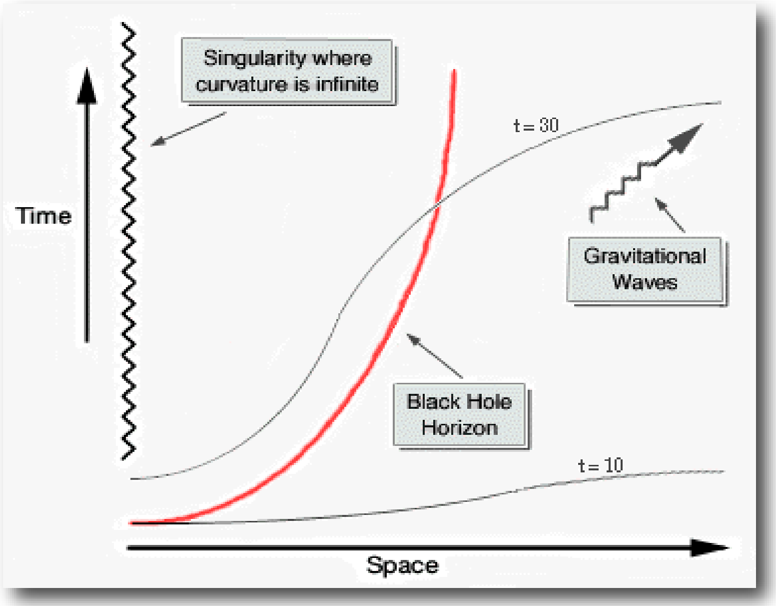
i) make a guess (i.e. prescribe a functional form) for the lapse, and shift and hope for the best: eg geodesic slicing $\alpha = 1\,,\ \beta^i = 0$

obviously not a good idea

Indeed the condition $\partial_t \alpha \sim -\alpha^2 f(\alpha) K$ represents a family of slicing conditions such that:

$$
\begin{aligned}
f = 0\,, (\alpha = 1 \text{ initially})\,, &\quad : \text{geodesic slicing} \\
f = 1\,, &\quad : \text{harmonic slicing} \\
f = 2/\alpha\,, &\quad : \text{"}1 + \log\text{" slicing} \\
f \to \infty\,, &\quad : \text{maximal slicing}
\end{aligned}
$$

The "1+log" slicing condition also has excellent singularity avoiding properties since $\partial_t \alpha \sim -\alpha$ and hence the lapse remains very small in those regions where it has "collapsed" to small values



Similarly, a popular choice for the shift is the hyperbolic "Gamma-driver" condition

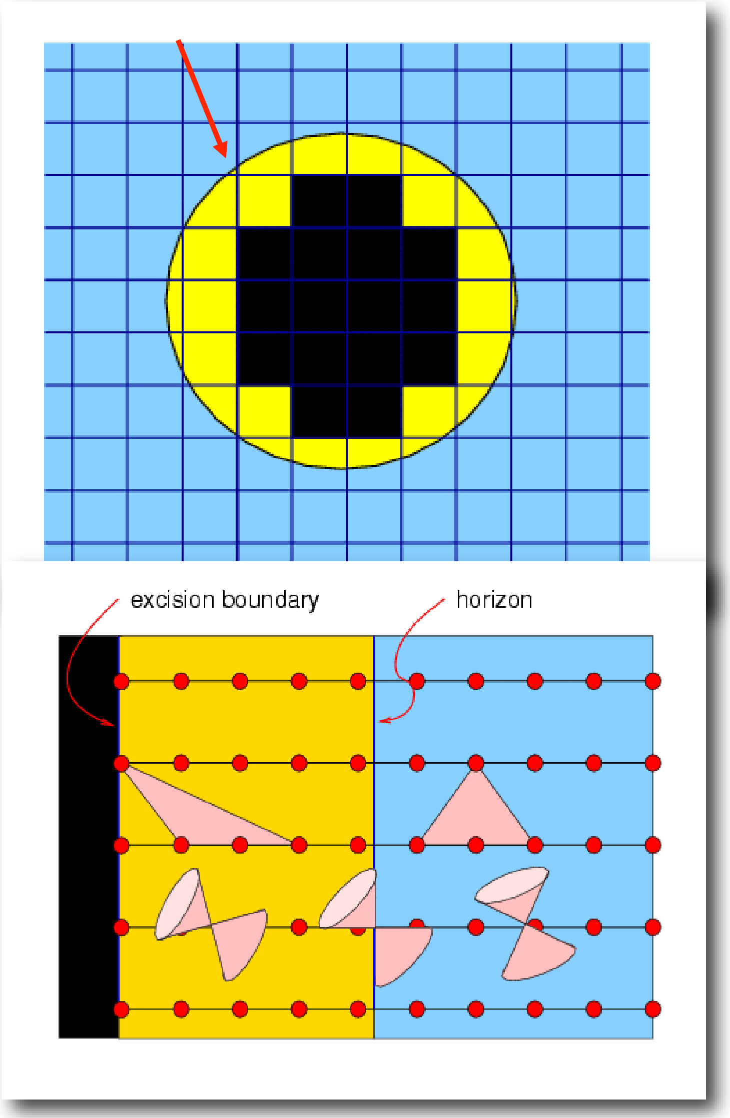
$$\begin{aligned}\partial_t \beta^i - \beta^j \partial_j \beta^i &= \frac{3}{4} \alpha B^i \,, \\ \partial_t B^i - \beta^j \partial_j B^i &= \partial_t \tilde{\Gamma}^i - \beta^j \partial_j \tilde{\Gamma}^i - \eta B^i \,,\end{aligned}$$

where η acts as a restoring force to avoid large oscillations in the shift and the driver tends to keep the Gammas constant (reminiscent of minimal distortion)

Overall, the "1+log" slicing condition and the "Gamma-driver" shift condition are the most widely used both in vacuum and non-vacuum spacetimes

excising parts of the spacetime with singularities…

apparent horizon found on a given Σ_t



Images by D. Pollney

In principle, the yellow region is causally disconnected from the blue one (ligth cones are "tilted in"); no boundary conditions would be needed at the apparent horizon.

In practice, the actual excision region (*"legosphere"*: black region) carved well inside the horizon.

NOTE:

- the Einstein equations are highly nonlinear in the yellow region! All sorts of numerical problems...
- the (apparent) horizon must be found; this is an expensive operation…
- the excised region has to move on the grid…

Extraction of gravitational waves

Weyl scalars

We then calculate Ψ_4 in terms of ADM related quantities as

$$\Psi_4 = C_{ij}\bar{m}^i\bar{m}^j,$$

where

$$C_{ij} \equiv R_{ij} - KK_{ij} + K_i{}^k K_{kj} - \mathrm{i}\epsilon_i{}^{kl}\nabla_l K_{jk}.$$

Then at a sufficiently large distance from the source the GWs in the two polarizations $h_\times, h_+$ can be written as

$$h_+ - \mathrm{i}h_\times = \lim_{r\to\infty}\int_0^t \mathrm{d}t' \int_0^{t'} \mathrm{d}t''\Psi_4$$

Then, eg, the projection of the momentum flux on the equatorial plane as

$$\mathcal{F}_i = \frac{\mathrm{d}P_i}{\mathrm{d}t} = \lim_{r\to\infty}\left\{\frac{r^2}{16\pi}\int d\Omega\, n_i \left|\int_{-\infty}^t dt\Psi_4\right|^2\right\}.$$

This quantity can be used, for instance, to calculate the recoil.

Recap

☑ The lapse and the shift have simple physical definitions and relate events on two different hypersurfaces.

☑ Getting a good formulation of the Einstein eqs will work only in conjunction with good gauge conditions. "1+log" slicing "Gamma-driver" conditions work well in a number of conditions.

☑ Even with suitable formulations and gauge conditions, any astrophysical prediction needs the calculation of "realistic" initial data and hence the solution of elliptic equations.

☑ GWs can be extracted with great accuracy. Several methods using either the radiative part of the Riemann tensor or perturbations of the Schwarzschild spacetime.

Recap (II)

- ☑ The ADM eqs are ill posed and not suitable for numerics.
- ☑ Alternative formulations (BSSNOK, CCZ4, Z4c) have been developed that are strongly hyperbolic and hence well-posed.
- ☑ Both formulations make use of the constraint equations and can use additional evolution equations to damp the violationsns
- ☑ The hyperbolic evolution eqs. to solve are: 6+6+(3+1+1) = 17. We also "compute" 3+1=4 elliptic constraint eqs

$$\mathcal{H} \equiv {}^{(3)}R + K^2 - K_{ij}K^{ij} = 0 \,, \quad \text{(Hamiltonian constraint)}$$

$$\mathcal{M}^i \equiv D_j(K^{ij} - g^{ij}K) = 0 \,, \quad \text{(momentum constraints)}$$

NOTE: these eqs are not solved but only monitored to verify

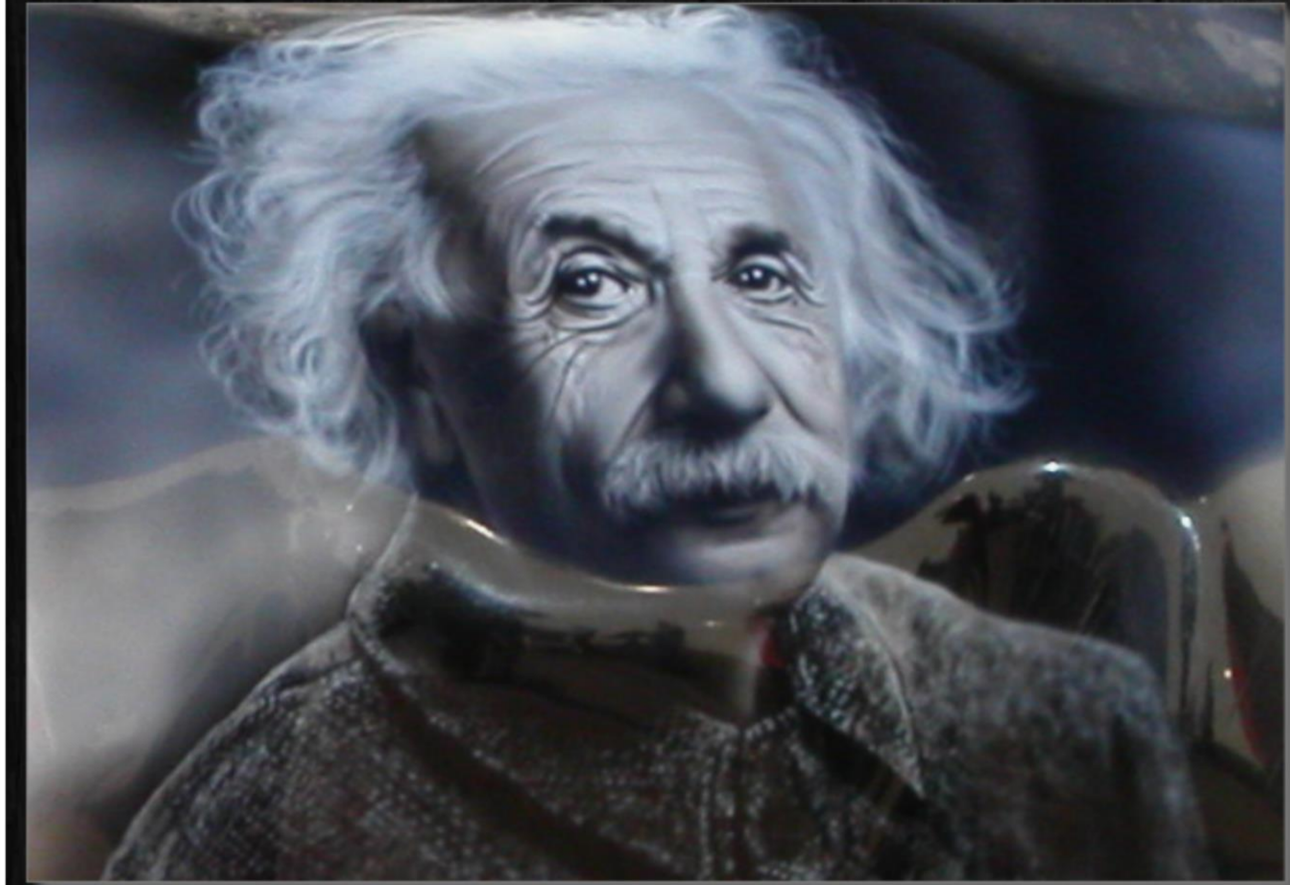
$$||\mathcal{H}|| \simeq ||\mathcal{M}^i|| < \varepsilon \sim 10^{-4} - 10^{-2}$$

General Relativity with the Computer

Allgemeine Relativiätstheorie mit dem Computer *von* Dr.phil.nat.Dr.rer.pol. Matthias Hanauske

Home Research Contact

Einführung | **Teil I** | **Teil II** | **Teil III** | *E-Learning*



Allgemeine Relativitätstheorie mit dem Computer
(General Theory of Relativity on the Computer)
Vorlesung SS 2016, Mo. 16-18.00 Uhr, PC-Pool 01.120

In dieser Vorlesung werden die mathematisch anspruchsvollen Gleichungen der Allgemeinen Relativitätstheorie (ART) in diversen Programmierumgebungen analysiert. Im ersten Teil des Kurses erlernen die Studierenden die Verwendung von Computeralgebra-Systemen (Maple und Mathematica). Die oft komplizierten und zeitaufwendigen Berechnungen der tensoriellen Gleichungen der ART können mit Hilfe dieser Programme erleichtert werden. Diverse Anwendungen der Einstein- und Geodätengleichung werden in Maple implementiert, quasi analytische Berechnungen durchgeführt und entsprechende Lösungen berechnet und visualisiert. Der zweite Teil des Kurses befasst sich mit der numerischen Berechnung von Neutronensternen und Weißen Zwergen mittels eines C/C++ Programms. Nach einer kurzen Auffrischung der grundlegenden Programmierkenntnisse, erstellen die Studierenden, gemeinsam mit dem Betreuer, ein Programm, das die Tolman-Oppenheimer-Volkov-Gleichung numerisch löst und visualisieren die Ergebnisse. Zusätzlich wird hierbei in die Grundkonzepte der parallelen Programmierung eingeführt und eine MPI- und OpenMP-Version des C/C++ Programms erstellt. Im dritten Teil des Kurses werden zeitabhängige numerische Simulationen der ART mittels des Einstein Toolkit durchgeführt und deren Ergebnisse mittels Python/Matplotlib visualisiert. Inhaltlich wird hierbei ebenfalls auf den, dem Programm zugrunde liegenden (3+1)-Split der ART eingegangen und, abhängig von den Vorkenntnissen der Studierenden, mehrere fortgeschrittene, astrophysikalisch relevante Probleme simuliert. Mögliche Themen dieses abschließenden Teils könnten die folgenden Systeme

http://itp.uni-frankfurt.de/~hanauske/VARTC/