# Experiments in Computer Simulations: Radial Oscillations of Relativistic Spherical Stars

*Pool Room 01.120  Johann Wolfgang Goethe-University  May 4, 2015*

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1. Introduction
2. Allgemeine Relativitätstheorie
3. Black Holes
4. Neutron Stars
5. Numerical General Relativity and Relativistic Hydrodynamics
6. The Einstein Toolkit
7. Radial Oscillations of Relativistic Spherical Stars

# Schedule

Schedule of the internship:

- 1.Day: Download and compile the Einstein Toolkit
- 1.Day: Run Simulations (3 Cowling, 1 Full)
- 1.Day: Visualizing data
- 1.Day: Calculate the Power Spectral Density (PSD, homework)
- 2.Day: PSD
- 2.Day: Visualizing data
- 2.Day: Questions

# Allgemeine Relativitätstheorie

Ästetikern oft als 'Schönste Gleichung der Physik' bezeichnet wird. Nach meiner Meinung liegt jedoch die eigentliche 'Schönheit' dieser Gleichung nicht in ihrer mathematischen Formulierung, sondern sie ist herausragend aufgrund der Idee selbst und aufgrund ihrer physikalischen Einfachheit:

$$\underbrace{R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu}}_{\text{Verformung der Raumzeit}} = \frac{8\pi\, G}{c^4} \underbrace{T_{\mu\nu}}_{\text{Energieformen des System}} \tag{2.1}$$

Die Einsteingleichung besagt, dass jegliche Energieformen die Raumzeit verändern; je mehr Energie sich an einem Ort befindet, desto mehr verbiegt sich die Raumzeit.

Die geniale Idee der ART ist nun, dass sie die anscheinenden Kräfte die durch die Raumzeitkrümmung auftreten, mit der Gravitationskraft gleichsetzt. In der ART wird die, zwischen zwei Körpern wirkende Gravitationskraft verursacht durch eine Raumzeitkrümmung.

# Allgemeine Relativitätstheorie

| | | |
|---|---|---|
| $\eta_{\mu\nu}$ | $\Rightarrow$ | $g_{\mu\nu}(x^\alpha)$ |
| Flache Raumzeit | | Gekrümmte Raumzeit |
| Kein Gravitationsfeld | | Gravitation |
| Inertialsystem | | Beschleunigtes System |

Wie bewegt sich nun ein Teilchen in einer durch Gravitationsfelder gekrümmten Raumzeit? Aufgrund des Hamilton'schen Prinzips der kleinsten Wirkung sollte das Teilchen den Weg des kleinsten Widerstandes wählen. Falls keine weiteren Kräfte auf das Teilchen einwirken, wird die realisierte Bahn des Teilchens die minimalste Strecke zwischen zwei Raumzeitpunkten $A$ und $B$ sein

$$\int_A^B ds = \int_A^B \sqrt{g_{\mu\nu} dx^\mu dx^\nu} \rightarrow \text{minimal!} \quad , \tag{2.8}$$

# Allgemeine Relativitätstheorie

bezeichnen kann. Parametrisiert man die Raumzeitkurve $x^\alpha$ mit einem affinen Parameter $\lambda$,[5] so muss die Weglänge bezüglich infinitesimaler Weg-Variationen $\delta_{x^\alpha(\lambda)}$ verschwinden

$$\delta_{x^\alpha(\lambda)} \int_A^B ds = \delta_{x^\mu(\alpha)} \int_A^B \sqrt{g_{\mu\nu} dx^\mu dx^\nu} \tag{2.9}$$

$$= \delta_{x^\alpha(\lambda)} \int_{\lambda_A}^{\lambda_B} \sqrt{g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}} d\lambda = 0 \quad . \tag{2.10}$$

Die dadurch entstehende Euler-Lagrangegleichung bezeichnet man als Geodätengleichung

$$\frac{d^2 x^\alpha}{d\lambda^2} = -\Gamma^\alpha_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} \quad , \tag{2.11}$$

wobei die Tensoren $\Gamma^\alpha_{\mu\nu}$ als Christoffelsymbole bzw. Konnexionen bezeichnet werden

$$\Gamma^\alpha_{\mu\nu} = \frac{g^{\alpha\lambda}}{2} \left( g_{\mu\nu|\lambda} + g_{\lambda\nu|\mu} + g_{\mu\lambda|\nu} \right) \quad . \tag{2.12}$$

Mit Hilfe der Geodätengleichung können wir also berechnen, wie sich Teilchen in vorgegebener gekrümmter Raumzeit bewegen müssen. Wie jedoch berech-

# Allgemeine Relativitätstheorie

Die Konnexionen bzw. Christoffelsymbole $\Gamma^{\mu}_{\beta\nu}$, welche sich aus der Metrik $g_{\mu\nu}$ und ihren ersten Ableitungen zusammensetzen, sind verantwortlich für die kovariante Ableitung $D_\mu$ in der ART:

$$D_\nu v^\mu := v^\mu{}_{||\nu} = v^\mu{}_{|\nu} + \Gamma^{\mu}_{\beta\nu} v^\beta \tag{B.4}$$

Der Riemanntensor $R_{\alpha\sigma\nu\mu}$ stellt die gravitative Feldstärke in der ART dar; er ergibt sich ebenfalls aus dem Kommutator der kovarianten Ableitungen:

$$\begin{aligned} [D_\mu, D_\nu]\, v^\alpha &= v^\alpha{}_{||\mu||\nu} - v^\alpha{}_{||\nu||\mu} \\ &= -\left(\Gamma^{\alpha}_{\sigma\nu|\mu} - \Gamma^{\alpha}_{\sigma\mu|\nu} + \Gamma^{\alpha}_{\beta\mu}\Gamma^{\beta}_{\sigma\nu} - \Gamma^{\alpha}_{\beta\nu}\Gamma^{\beta}_{\sigma\mu}\right) v^\sigma \\ &=: -R^\alpha{}_{\sigma\nu\mu} v^\sigma \end{aligned} \tag{B.5}$$

# Black Holes

**Schwarzschild Lösung**

Wir betrachten im folgenden den Spezialfall eines kugelsymmetrischen Gravitationsfeldes. Die zugehörige Raumzeitmetrik $g_{\mu\nu}(x^\alpha)$ und das Weglängenelement ds müssen dann ebenfalls Kugelsymmetrie besitzen, was bedeutet, dass sie bei gleicher Entfernung vom Ursprung in allen Raumzeitpunkten gleiche Werte besitzen sollen. Verwenden wir räumliche Kugelkoordinaten $(x^\mu = (ct, x, y, z) \rightarrow x^\mu = (ct, r, \theta, \phi))$ so ist die allgemeinste Form der Metrik die folgende [5]:

$$g_{\mu\nu} = \begin{pmatrix} l(r,t) & a(r,t) & 0 & 0 \\ a(r,t) & h(r,t) & 0 & 0 \\ 0 & 0 & k(r,t) & 0 \\ 0 & 0 & 0 & k(r,t)\sin^2\theta \end{pmatrix} \tag{2.25}$$

# Black Holes

$$g_{\mu\nu} = \begin{pmatrix} e^{\nu(r,t)} & 0 & 0 & 0 \\ 0 & -e^{\lambda(r,t)} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2 \sin^2\theta \end{pmatrix} \tag{2.26}$$

$$ds^2 = c^2 e^{\nu(r,t)} dt^2 - e^{\lambda(r,t)} dr^2 - r^2 \left(\sin^2\theta d\phi^2 + d\theta^2\right) \tag{2.27}$$

Die Masse eines kugelsymmetrischen schwarzen Loches befindet sich (wie wir im folgenden sehen werden) konzentriert im Ursprungspunkt bei $r = 0$, so dass der gesamte Raum (ohne den Punkt $r = 0$) materiefrei ist. Der Energieimpulstensor verschwindet demnach im Außenraum identisch $T_{\mu\nu} \equiv 0$, so dass sich die Einsteingleichung (Gl. 2.23) wie folgt vereinfacht:

$$R^{\mu\nu} - \frac{1}{2} R\, g^{\mu\nu} = 8\pi\kappa T^{\mu\nu} = 0$$

$$\Rightarrow \qquad R^{\mu\nu} = 0 \tag{2.28}$$

# Black Holes

wird als Schwarzschildradius bezeichnet

$$R_S = \frac{2G\,M}{c^2} \quad . \tag{2.36}$$

Die Schwarzschildmetrik und das zugehörige Weglängenelement nimmt nun die folgende Form an

$$g_{\mu\nu} = \begin{pmatrix} \left(1-\frac{R_S}{r}\right) & 0 & 0 & 0 \\ 0 & -\left(1-\frac{R_S}{r}\right)^{-1} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2\sin^2\theta \end{pmatrix} \tag{2.37}$$

$$ds^2 = c^2\left(1-\frac{R_S}{r}\right)dt^2 - \left(1-\frac{R_S}{r}\right)^{-1}dr^2 - r^2\left(\sin^2\theta d\phi^2 + d\theta^2\right) \ .$$

# Black Holes

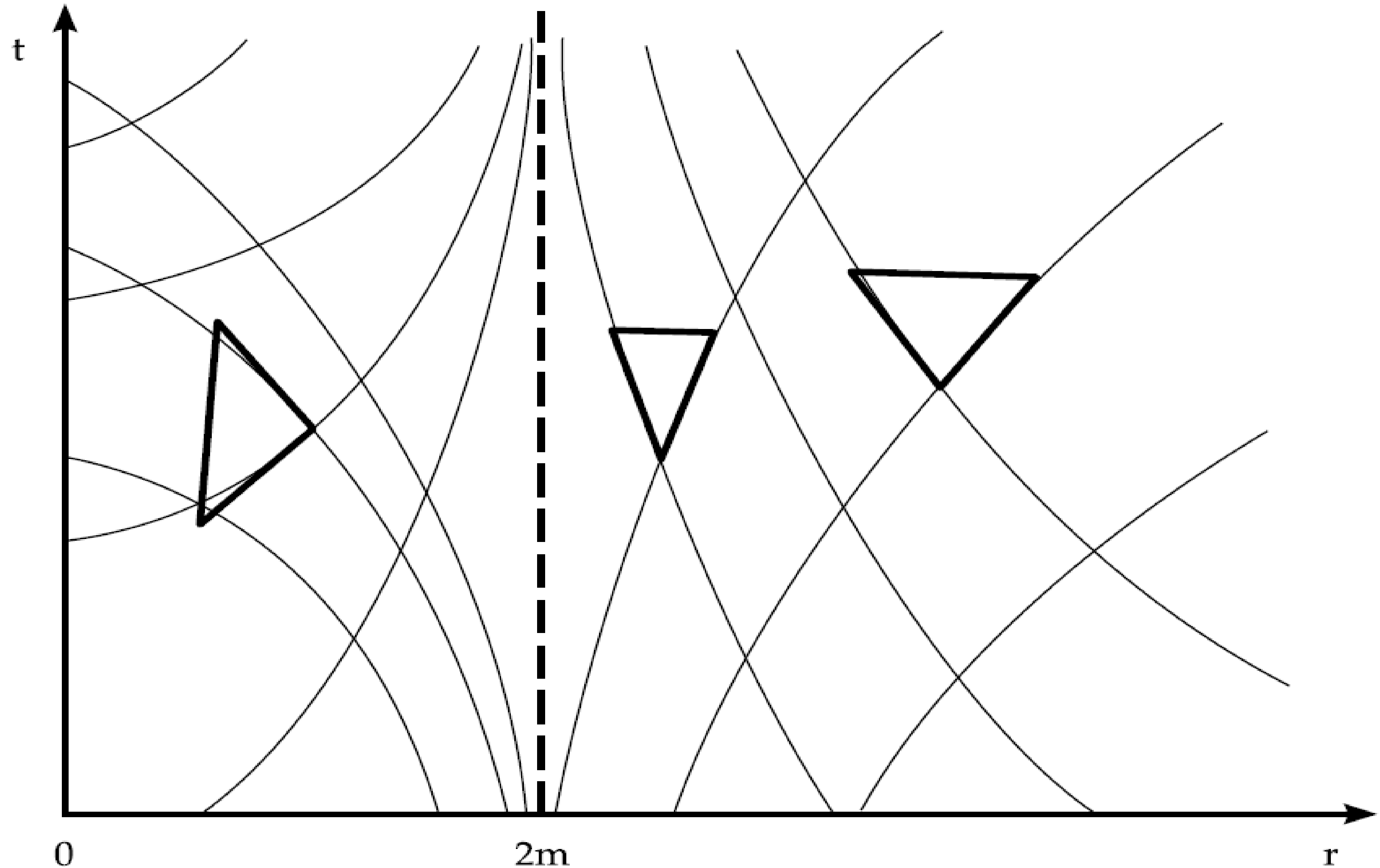



Abbildung 2.1: Raumzeitdiagramm der Schwarzschildmetrik in Schwarzschildkoordinaten.

# Black Holes

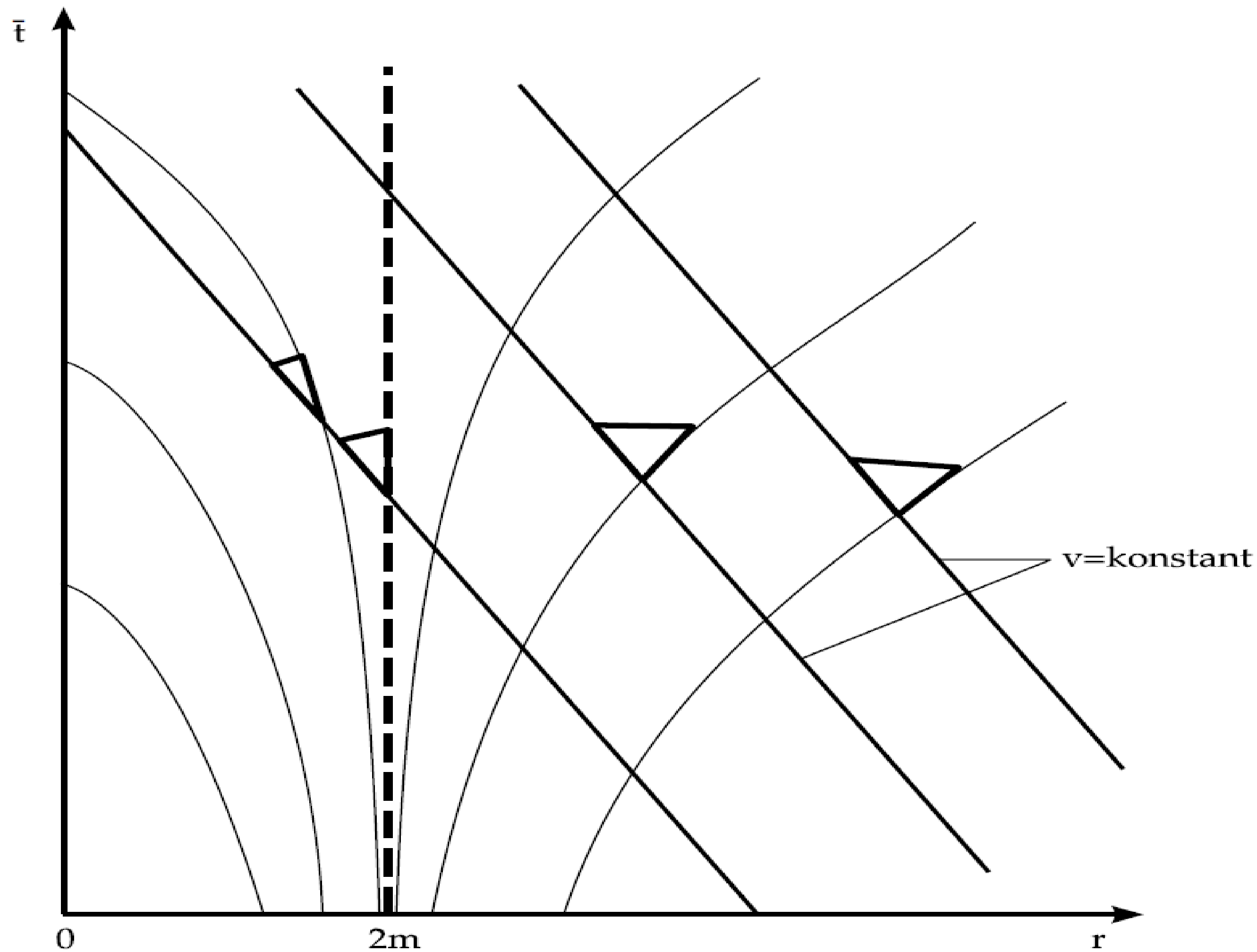



Abbildung 2.2: Raumzeitdiagramm der Schwarzschildmetrik in avancierten Eddington-Finkelstein Koordinaten.

# Neutron Stars

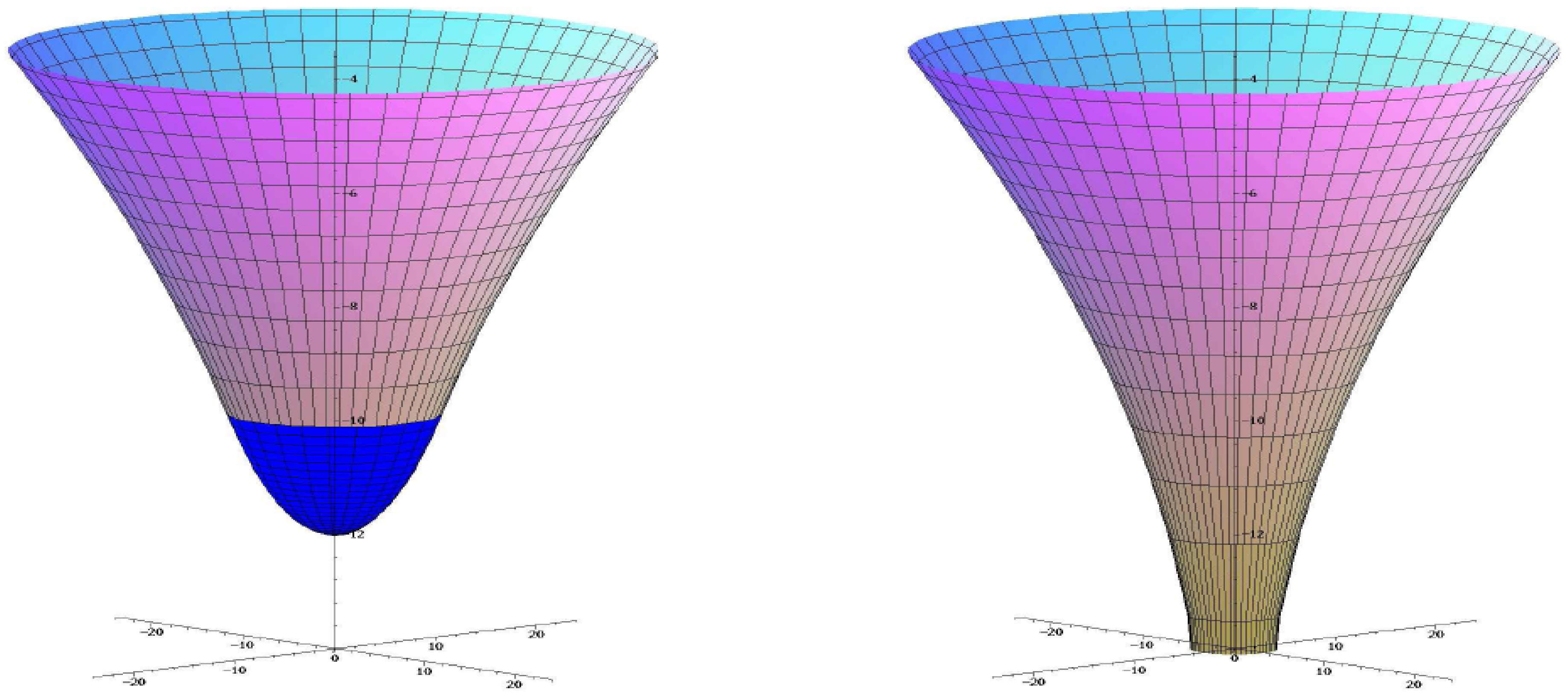
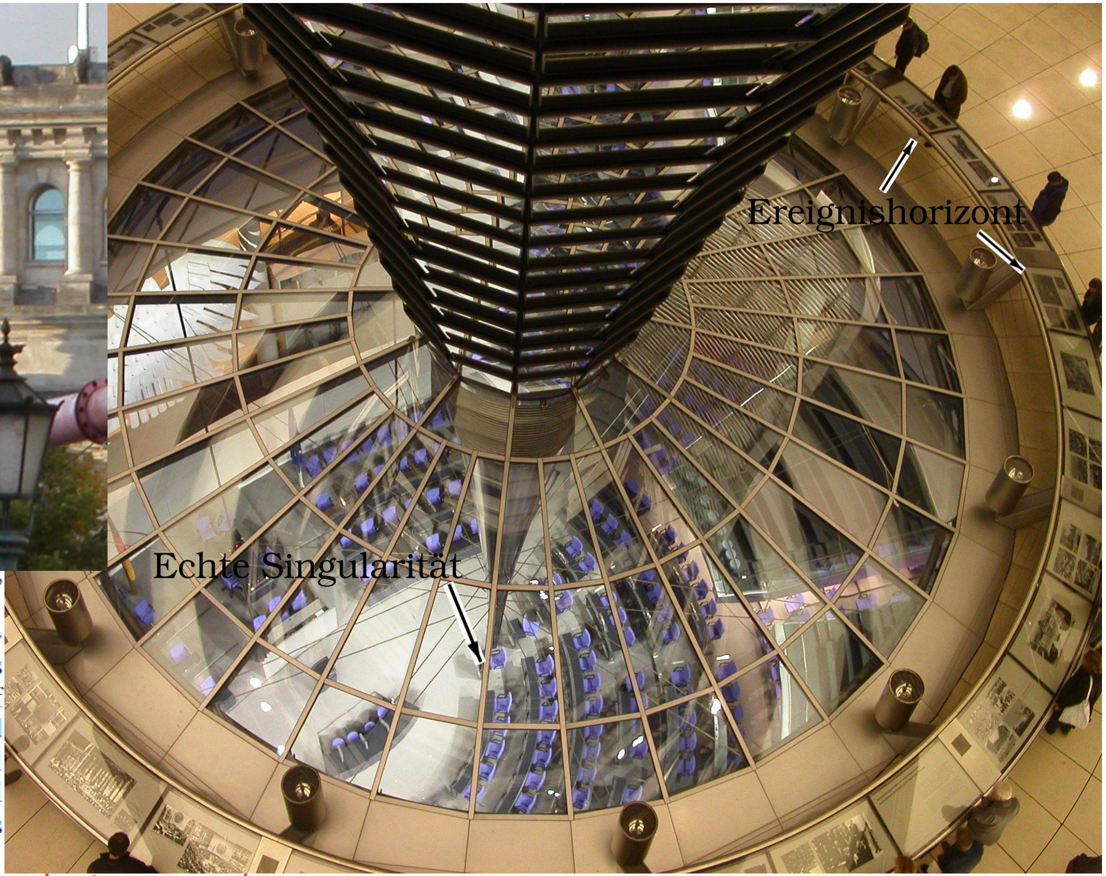


Abbildung 2.5: Eingebettetes Raumzeitdiagramm eines Neutronensterns (links) und eines schwarzen Loches (rechts) wobei $M = 1.4\,M_\odot$ und die x- und y-Achse in Einheiten km dargestellt sind.

# Black holes and the German Reichstag




the subject.

Unfortunately, modern physics is impossible to comprehend using intuition alone. How can bizarre concepts such as the curvature of space–time or the event horizon of a black hole be understood? What possible imagery could help non-scientists to grasp the significance and vital importance of some of the major insights of theoretical physics? Finding a simple way of conveying those ideas seemed an impossible task.

Lost in thought, I looked up and realized I had almost reached my desired destination as the modern glass dome

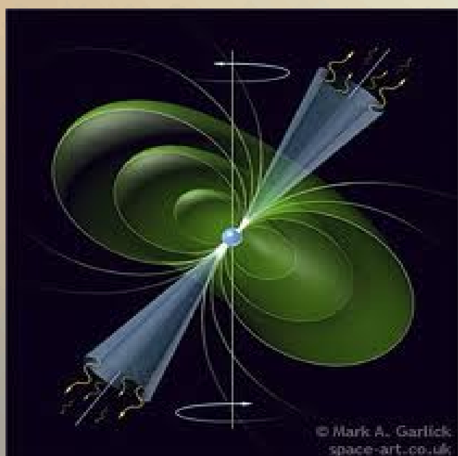
diagrams used to illustrate

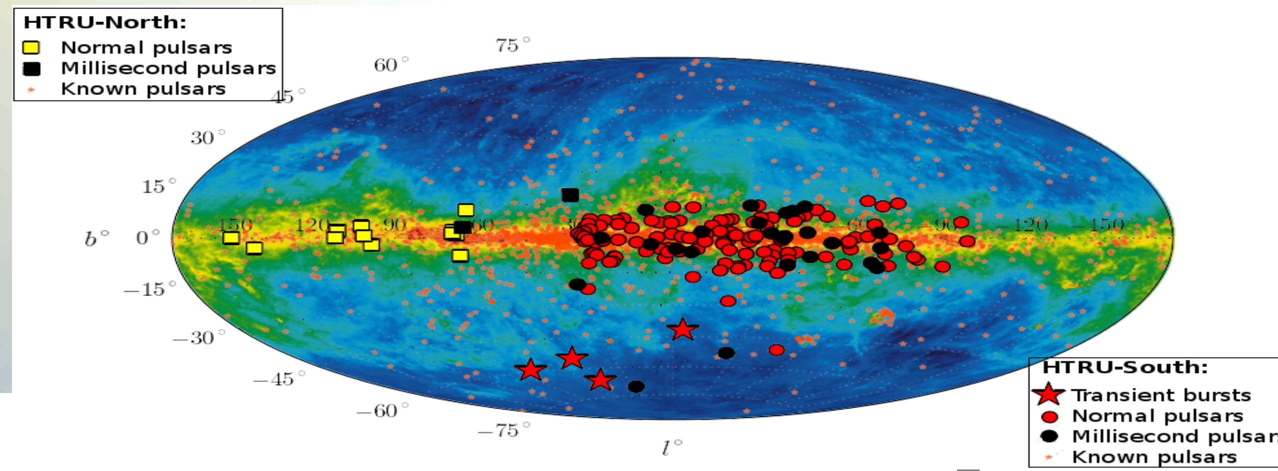
sion of the Nazi era.

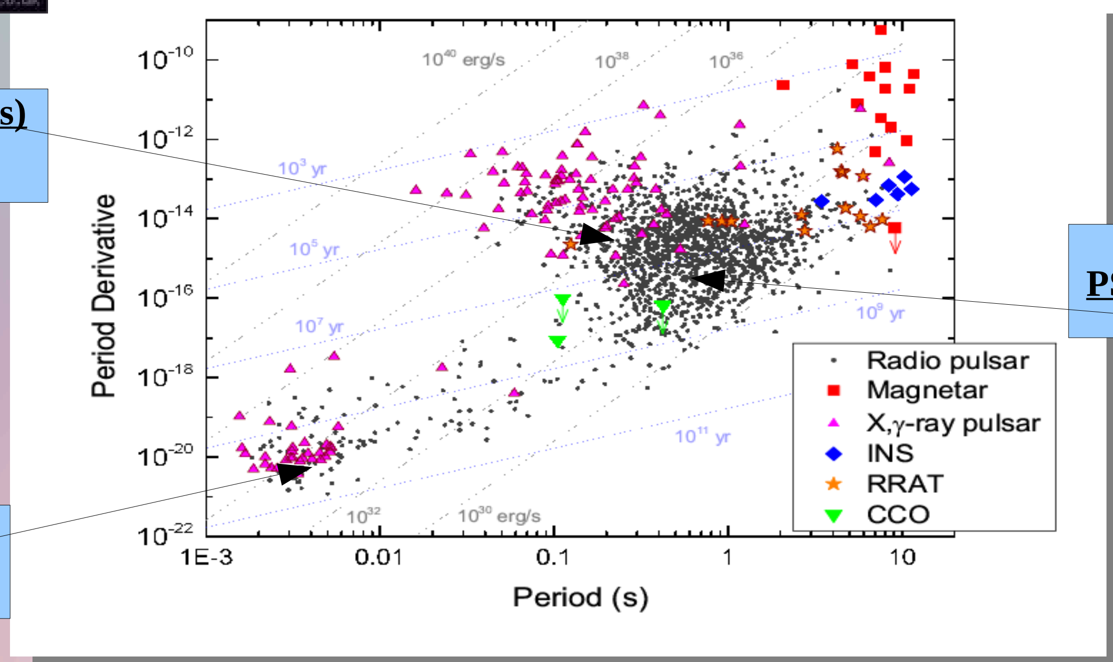
Suddenly I saw the significance of the information

See http://wiap.wiwi.uni-frankfurt.de/Publications/LateralThoughts.pdf
http://fias.uni-frankfurt.de/~hanauske/new/eqgt/HSK_2011/Einfuerung/html/mov/Marathonvorlesung2003.mov

# Pulsars ⟷ Neutron Stars




**PSR B0531+21 (33.5 ms)**

**Crab Pulsar**

**PSR B0329+54 (0.715 s)**

**PSR B1937+21 (1.56 ms)**

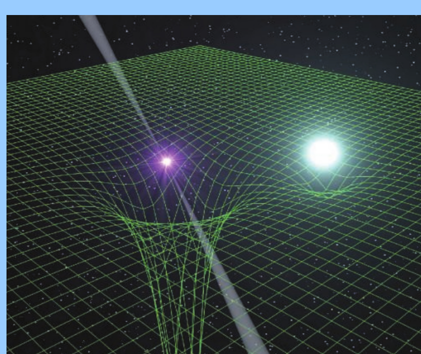
# Observed Masses of Compact Star Binaries

**PSR J1906+0746** Van Leeuwen et al, arXiv:1411.1518

144-ms pulsar, discovered in in 2004
Orbital period: 3.98 hours, Eccentricity: 0.085
Pulsar mass: 1.291(11), Companion mass 1.322(11)
Observed between 1998-2009,
then it disappeared due to spin precession

**Double Pulsar PSR J0737-3039**

Orbital period: 147 min, Eccentricity: 0.088
pulsar A: P=23 ms, M=1.3381(7)
pulsar B: P=2.7 s, M=1.2489(7)
Pulsar A is eclipsed once per orbit by B (for 30 s)
Kramer, Wex, Class. Quantum Grav. 2009



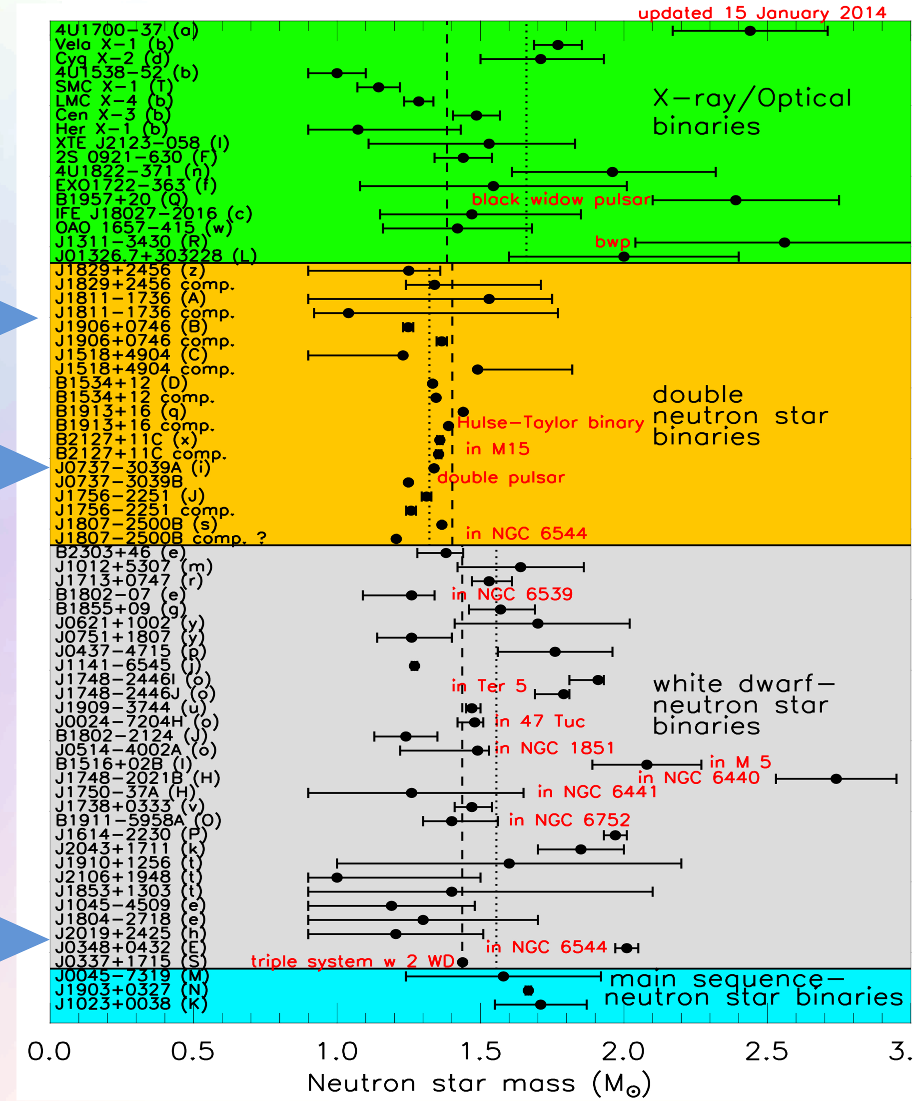
**PSR J0348+0432**

Orbital Period:
2.46 hours

Pulsar mass:
2.01+-0.04

white dwarf mass:
0.172+-0.003

Picture from J. Antoniadis et.al. Science 2013

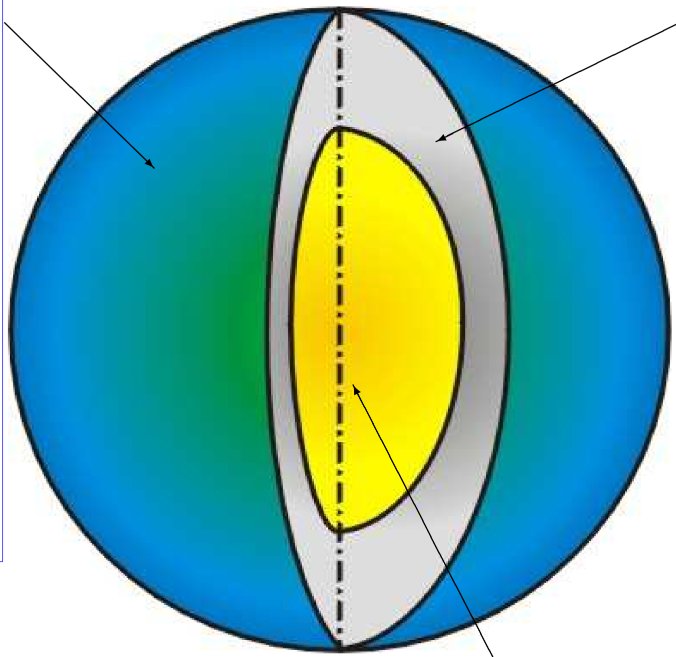


# Summary: Neutron Stars

## Schematic Structure of a Neutron Star

### Outer Envelopes

The outer envelopes of a neutron star constist of a thin plasma atmosphere where the thermal radiation is formed and a outer and inner crust which consist of electrons and nuclei. The whole outer envelope is about one kilometer thick and it occupies the density range $\epsilon \leq 0.5\,\epsilon_0$ .



### Outer Core

The outer core consists mainly of neutrons with several per cent admixture of protons, electrons and myons. It is several kilometers thick and occupies the density range $0.5\,\epsilon_0 < \epsilon \leq 2\,\epsilon_0$ .

Nuclear matter density:
$\epsilon_0 = 2.8 \cdot 10^{14}$ g/cm$^3$

### Inner Core

In the inner core of a neutron star hyperonic particles $(\Sigma^-, \Lambda, \Xi ...)$ are present. The inner core extends to the center of the star where its central density can be as high as $\epsilon \approx 15\,\epsilon_0$ .

# General Relativity and Quantum Cromodynamics

| ART | Yang-Mills-Theories |
|---|---|
| $D_\beta v^\alpha = \partial_\beta v^\alpha + \Gamma^\alpha_{\sigma\beta} v^\sigma$ | $D_{\beta a}{}^b = \partial_\beta 1_a{}^b + ig\, A_{\beta a}{}^b$ |
| $R^\delta{}_{\mu\alpha\beta} v^\mu = [D_\alpha, D_\beta] v^\delta$ <br> $R^\delta{}_{\mu\alpha\beta} = \Gamma^\delta_{\mu\alpha\|\beta} - \Gamma^\delta_{\mu\beta\|\alpha}$ <br> $+\Gamma^\delta_{\nu\beta}\Gamma^\nu_{\mu\alpha} + \Gamma^\delta_{\nu\alpha}\Gamma^\nu_{\mu\beta}$ | $F_{\alpha\beta a}{}^b = \frac{1}{ig}\left[D_{\alpha a}{}^c, D_{\beta c}{}^b\right]$ <br> $= A_{\beta a}{}^b{}_{\|\alpha} - A_{\alpha a}{}^b{}_{\|\beta}$ <br> $+\frac{1}{ig}\left[A_{\alpha a}{}^c, A_{\beta c}{}^b\right]$ |
| $\mathcal{L}_G = R + \underbrace{(c_1 R_{\mu\nu}R^{\mu\nu} + ...)}_{\equiv 0 \text{ for ART}}$ | $\mathcal{L}_{YM} = \frac{1}{4} F_{\mu\nu a}{}^b F^{\mu\nu}{}_a{}^b$ |

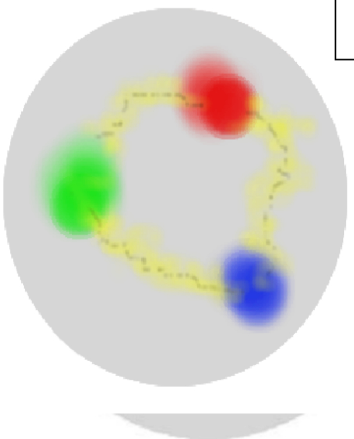
QuantumCromoDynamic:

$(SU(3)_{(c)}$- Color Yang-Mills-Gauge Theory)

$D_{\beta A}{}^B = \partial_\beta 1_A{}^B + ig\, G_{\beta A}{}^B$

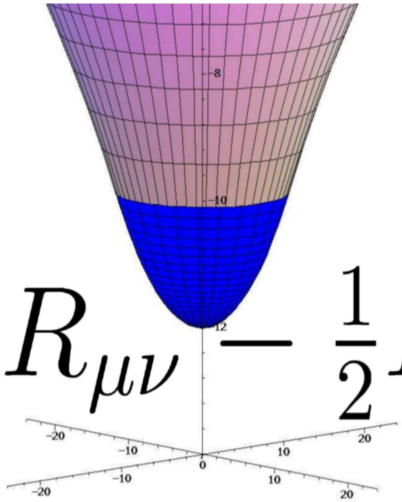
$_A, {}^B$ =red, green, blue

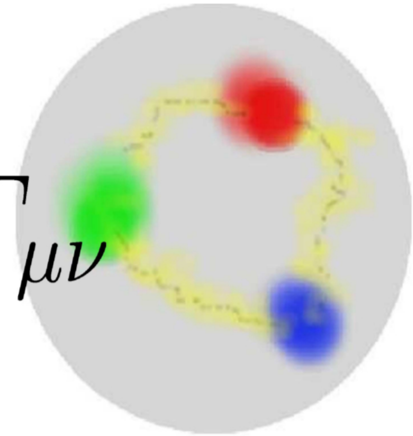
$$\psi^f_A = \begin{pmatrix} \psi^f_r \\ \psi^f_g \\ \psi^f_b \end{pmatrix}$$



Confinement

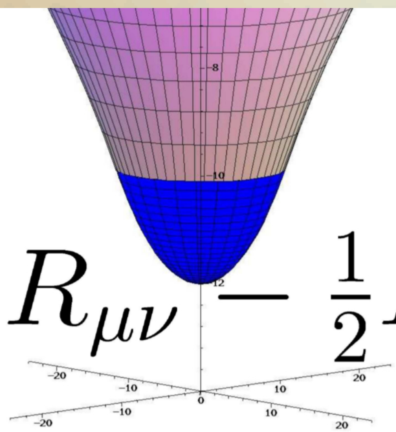
chiral symmetry, ...



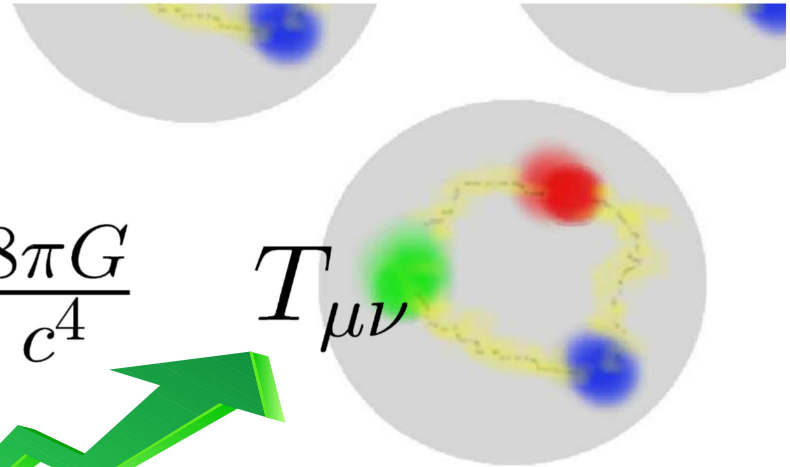



$$R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

# Neutron Stars (NS)



$$R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$



Relativistic Mean-Field Hadronic Models

$$\sum_B \quad (p, n, \Lambda, \Sigma^-, \Sigma^0, \Sigma^+, \Xi^-, \Xi^0)$$

$$\begin{aligned} \mathcal{L} &= \sum_B \overline{\psi}_B \left(i\partial\!\!\!/ - m_B\right) \psi_B + \frac{1}{2}\partial^\mu \sigma \partial_\mu \sigma - \frac{1}{2} m_\sigma^2 \sigma^2 - \frac{a}{3}\sigma^3 - \frac{b}{4}\sigma^4 - \frac{1}{4}\omega^{\mu\nu}\omega_{\mu\nu} \\ &+ \frac{1}{2} m_\omega^2 \omega^\mu \omega_\mu - \frac{1}{4}\vec{\rho}^{\mu\nu}\vec{\rho}_{\mu\nu} + \frac{1}{2} m_\rho^2 \vec{\rho}^\mu \vec{\rho}_\mu + \sum_B \overline{\psi}_B \left(g_{\sigma B}\sigma + g_{\omega B}\omega^\mu \gamma_\mu + g_\rho \vec{\rho}^\mu \gamma_\mu \vec{\tau}_B\right) \psi_B \end{aligned}$$

$$\begin{aligned} \mathcal{L}^{YY} &= \frac{1}{2}\left(\partial^\mu \sigma^* \partial_\mu \sigma^* - m_{\sigma^*}^2 \sigma^{*2}\right) - \frac{1}{4}\phi^{\mu\nu}\phi_{\mu\nu} + \frac{1}{2} m_\phi^2 \phi^\mu \phi_\mu \\ &+ \sum_Y \overline{\psi}_Y \left(g_{\sigma^* Y}\sigma^* + g_{\phi Y}\phi^\mu \gamma_\mu\right) \psi_Y \quad , \end{aligned}$$

$$\mathcal{L}_{\text{lep}} = \sum_{l=e,\mu} \overline{\psi}_l [i\gamma_\mu \partial^\mu - m_l] \psi_l$$

# Neutron Stars

$$g_{\mu\nu} = \begin{pmatrix} e^{\nu(r)} & 0 & 0 & 0 \\ 0 & -e^{\lambda(r)} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2 \sin^2\theta \end{pmatrix} \quad . \tag{2.45}$$

Das Einsetzen dieses Ansatzes der Metrik in die Einsteingleichung

$$G^{\mu}{}_{\nu} = R^{\mu}{}_{\nu} - \frac{1}{2} R\, g^{\mu}{}_{\nu} = 8\pi\kappa T^{\mu}{}_{\nu} \tag{2.46}$$

liefert das folgende System von Differentialgleichungen:

$$\begin{aligned} G^{t}{}_{t} = & \quad -e^{-\lambda}\left(\frac{1}{r^2} - \frac{\lambda^{|}}{r}\right) + \frac{1}{r^2} && = 8\pi\kappa T^{t}{}_{t} \\ G^{r}{}_{r} = & \quad -e^{-\lambda}\left(\frac{1}{r^2} + \frac{\nu^{|}}{r}\right) + \frac{1}{r^2} && = 8\pi\kappa T^{r}{}_{r} \\ G^{\theta}{}_{\theta} = & \quad -\frac{e^{-\lambda}}{2}\left(\nu^{||} - \frac{\lambda^{|}\nu^{|}}{2} + \frac{(\nu^{|})^2}{2} + \frac{\nu^{|}-\lambda^{|}}{r}\right) && = 8\pi\kappa T^{\theta}{}_{\theta} \\ G^{\phi}{}_{\phi} = & \quad G^{\theta}{}_{\theta} && = 8\pi\kappa T^{\phi}{}_{\phi} \end{aligned} \tag{2.47}$$

# Neutron Stars

$1..3, i \neq j$) vernachlässigen. Der Energieimpulstensor $T^{\mu\nu}$ einer solchen idealen Flüssigkeit, lokal betrachtet an seinem Ort, kann wie folgt geschrieben werden

$$T^{\mu\nu} = (\epsilon + P)u^\mu u^\nu - g^{\mu\nu} P \quad \text{mit: } u^\mu = \frac{dx^\mu}{d\tau} \quad , \tag{2.48}$$

wobei $u^\mu$ die 4er Geschwindigkeit der Materie ist, $\tau$ die lokale Eigenzeit an einem betrachteten Materiepunkt beschreibt ($d\tau = \sqrt{ds^2} = \sqrt{g_{tt}}\, dt$, $t$ ist die Koordinatenzeit eines unendlich entfernten Beobachters), $\epsilon$ die Energiedichte und $P$ der Druck der Materie ist.

# Neutron Stars

als die **Tollman-Oppenheimer-Volkoff (TOV) Gleichungen**

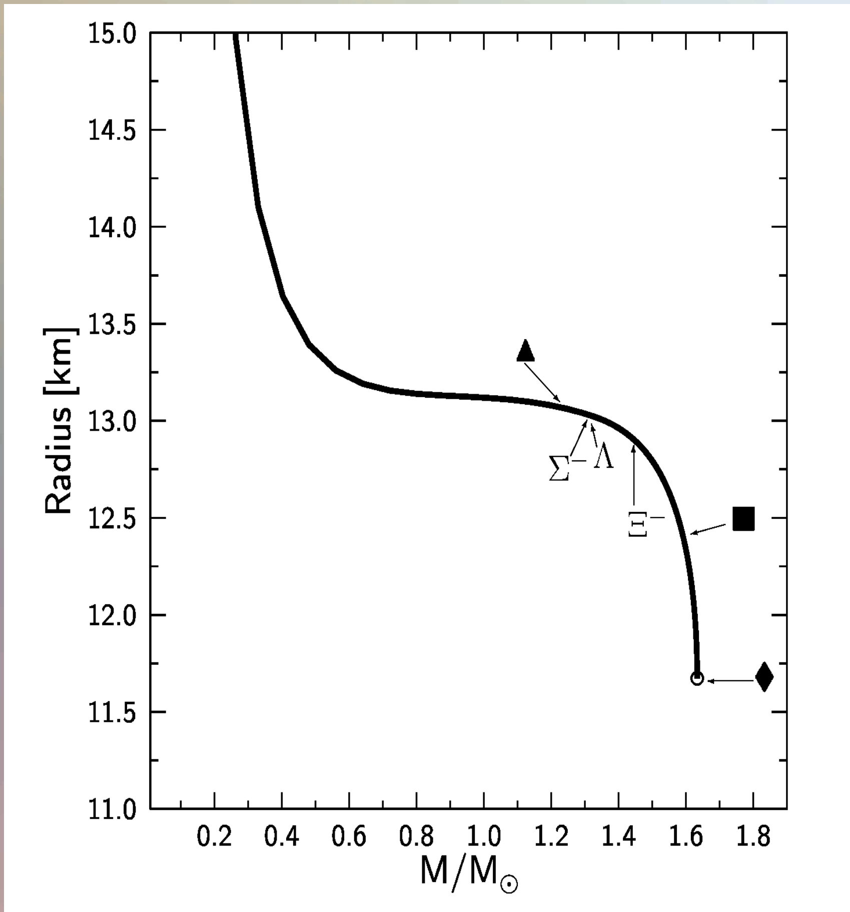
$$\begin{aligned} \frac{dP}{dr} &= -\frac{(\epsilon + P)4\pi r^3 + m}{r(r - 2\,m)} \\ m(r) &= \int_0^r 4\pi \tilde{r}^2 \epsilon(\tilde{r})\; d\tilde{r} \\ \frac{d\nu}{dr} &= \frac{8\pi P r^3 + 2\,m}{r(r - 2\,m)} \quad , \end{aligned} \tag{2.61}$$

wobei die raumzeitliche Struktur durch die folgenden Ausdrücke bestimmt ist

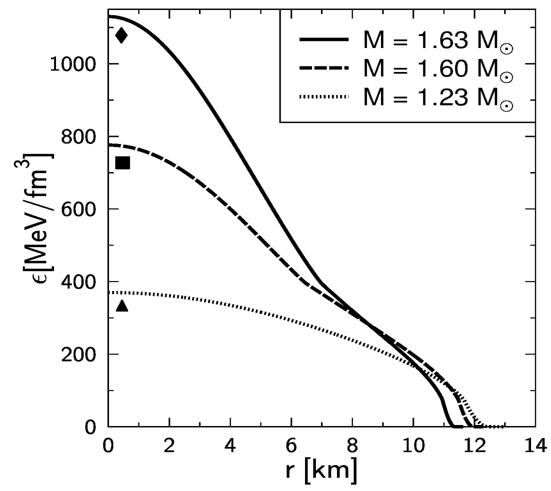
$$\begin{aligned} g_{\mu\nu} &= \begin{pmatrix} e^{\nu(r)} & 0 & 0 & 0 \\ 0 & -\left(1 - \frac{2\,m(r)}{r}\right)^{-1} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2 \sin^2\theta \end{pmatrix} \\ ds^2 &= e^{\nu(r)} dt^2 + \left(1 - \frac{2\,m(r)}{r}\right)^{-1} dr^2 - r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right) \; . \end{aligned}$$

# Neutron Star Properties

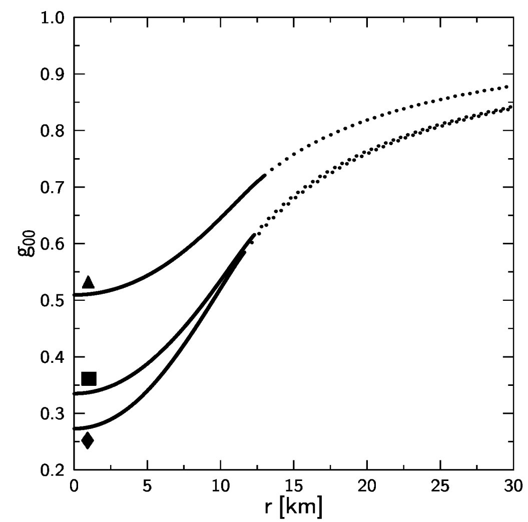
The neutron star radius as a function of its mass. A low, middle and high density star is displayed within the figure. Additionally the onset of hyperonic particles is visualized.




Energy density profiles of three neutron stars with different central densities and masses. The low density stars do not contain any hyperons, whereas the other two stars do have hyperons in their inner core.

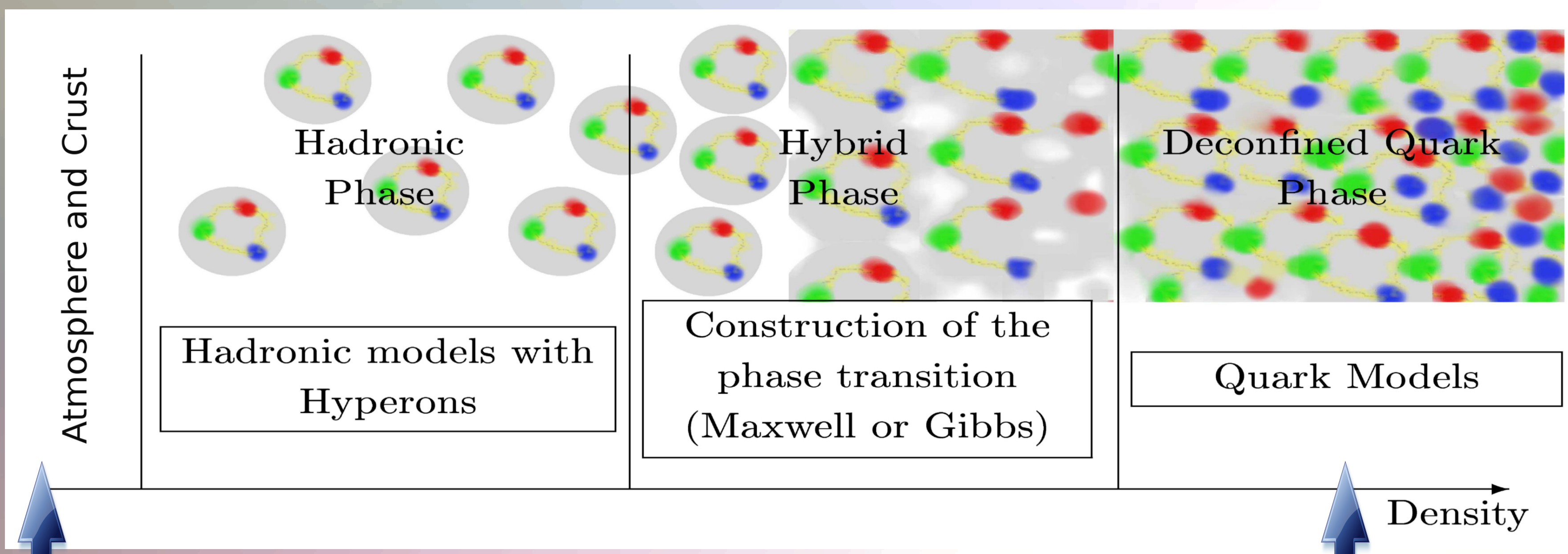



Time-time component of the metric tensor as a function of the radial coordinate. The solid line corresponds to the inner TOV-solution, whereas the doted curve depicts the outer Schwarzschild part.



# The QCD – Phase Transition

The appearance of the QCD - phase transition (the transition from confined hadronic to deconfined quark matter) will change the properties of neutron stars. Whether this change will be visible with telescopes and gravitational wave antennas depends strongly on the equation of state of hadronic and quark matter and on the construction of the phase transition.

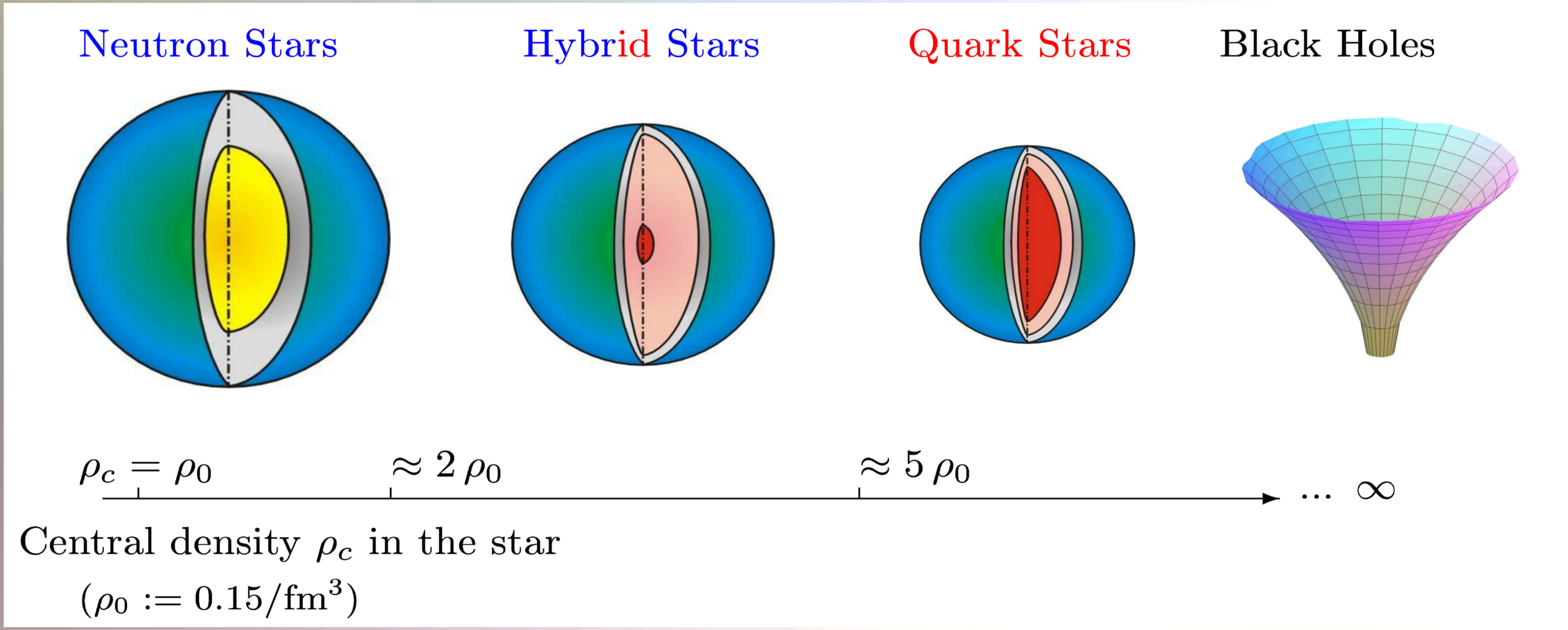



Surface of the Star

Center of the Star

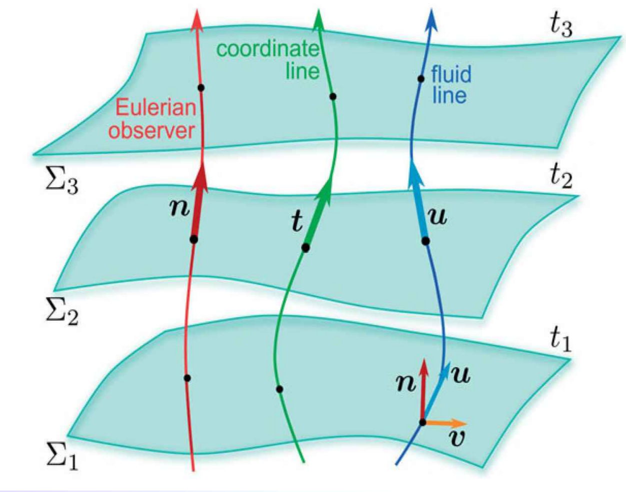
# The Compact Star Zoo

Depending on the model used, the compact star zoo consists of different inhabitants: e.g. neutron stars with and without hyperons, quark stars and strange quark stars, hybrid stars with color superconducting quark matter, hybrid stars with Bose-Einstein condensates of antikaons.



# Relativistic Hydrodynamics and Numerical General Relativity

A realistic numerical simulation of a twin star collapse, a merger of two compact stars or a collapse to a black hole, needs to go beyond a static, spherically symmetric TOV-solution of the Einstein- and Hydrodynamical equations.

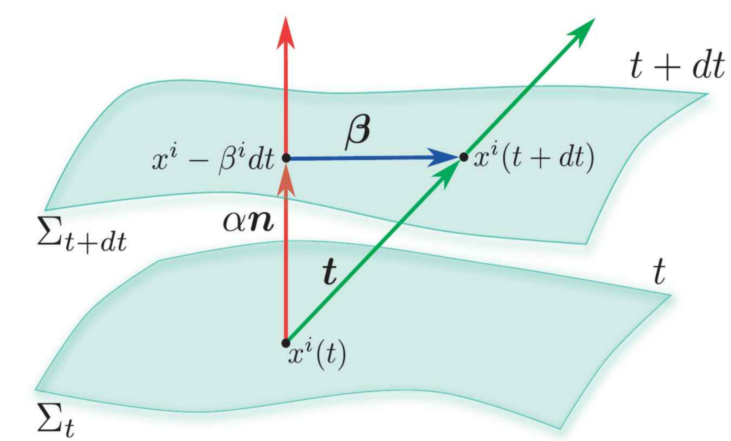



$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi T_{\mu\nu}$$

$$\nabla_\mu (\rho u^\mu) = 0 \,, \qquad \nabla_\nu T^{\mu\nu} = 0 \,.$$

$$g_{\mu\nu} = \begin{pmatrix} -\alpha^2 + \beta_i \beta^i & \beta_i \\ \beta_i & \gamma_{ij} \end{pmatrix}$$

(3+1) decomposition of spacetime




$$d\tau^2 = \alpha^2 (t, x^j) dt^2$$

$$x^i_{t+dt} = x^i_t - \beta^i (t, x^j) dt$$

All figures and equations from: Luciano Rezzolla, Olindo Zanotti: Relativistic Hydrodynamics, Oxford Univ. Press, Oxford (2013)

# The ADM equations

The ADM (Arnowitt, Deser, Misner) equations come from a reformulation of the Einstein equation using the (3+1) decomposition of spacetime.

$$\partial_t \gamma_{ij} = -2\alpha K_{ij} + \mathscr{L}_{\boldsymbol{\beta}}\gamma_{ij} = -2\alpha K_{ij} + D_i\beta_j + D_j\beta_i$$

$$\partial_t K_{ij} = -D_iD_j\alpha + \beta^k\partial_k K_{ij} + K_{ik}\partial_j\beta^k + K_{kj}\partial_i\beta^k + \alpha\left({}^{(3)}R_{ij} + KK_{ij} - 2K_{ik}K^k{}_j\right) + 4\pi\alpha\left[\gamma_{ij}\left(S-E\right) - 2S_{ij}\right)]$$

Time evolving part of ADM

$$D_j(K^{ij} - \gamma^{ij}K) = 8\pi S^i \qquad {}^{(3)}R + K^2 - K_{ij}K^{ij} = 16\pi E$$

Constraints on each hypersurface

Three dimensional covariant derivative

$$D_\nu := \gamma^\mu{}_\nu \nabla_\mu = (\delta^\mu_\nu + n_\nu n^\mu)\nabla_\mu$$

Three dimensional Riemann tensor

$${}^{(3)}R^\mu{}_{\nu\kappa\sigma} = \partial_\kappa\,{}^{(3)}\Gamma^\mu_{\nu\sigma} - \partial_\sigma\,{}^{(3)}\Gamma^\mu_{\nu\kappa} + {}^{(3)}\Gamma^\mu_{\lambda\kappa}\,{}^{(3)}\Gamma^\lambda_{\nu\sigma} - {}^{(3)}\Gamma^\mu_{\lambda\sigma}\,{}^{(3)}\Gamma^\lambda_{\nu\kappa}$$

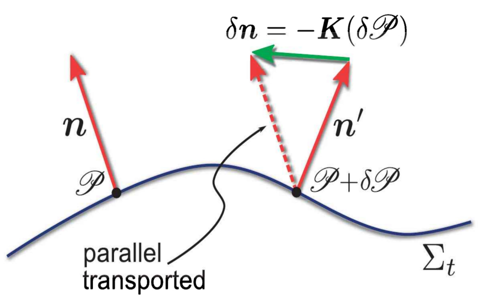
$${}^{(3)}\Gamma^\alpha_{\beta\gamma} = \frac{1}{2}\gamma^{\alpha\delta}\left(\partial_\beta\gamma_{\gamma\delta} + \partial_\gamma\gamma_{\delta\beta} - \partial_\delta\gamma_{\beta\gamma}\right)$$

Spatial and normal projections of the energy-momentum tensor:

$$S_{\mu\nu} := \gamma^\alpha{}_\mu\,\gamma^\beta{}_\nu T_{\alpha\beta}\,,$$
$$S_\mu := -\gamma^\alpha{}_\mu\, n^\beta T_{\alpha\beta}\,,$$
$$S := S^\mu{}_\mu\,,$$
$$E := n^\alpha\, n^\beta T_{\alpha\beta}\,,$$

Extrinsic Curvature:

$$K_{\mu\nu} := -\gamma^\lambda{}_\mu \nabla_\lambda n_\nu$$

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# From ADM to BSSNOK

Unfortunately the ADM equations are only weakly hyperbolic (mixed derivatives in the three dimensional Ricci tensor) and therefore not "well posed". It can be shown that by using a conformal traceless transformation, the ADM equations can be written in a hyperbolic form. This reformulation of the ADM equations is known as the BSSNOK (Baumgarte, Shapiro, Shibata, Nakamuro, Oohara, Kojima) formulation of the Einstein equation. Most of the numerical codes use this (or the CCZ4) formulation.

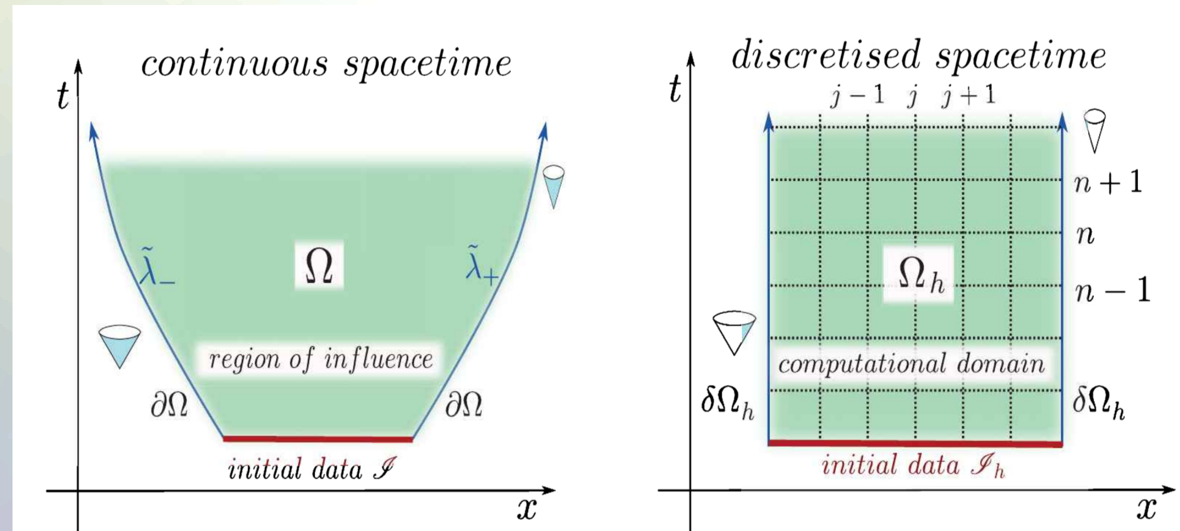
# The 3+1 Valencia Formulation of the Relativistic Hydrodynamic Equations

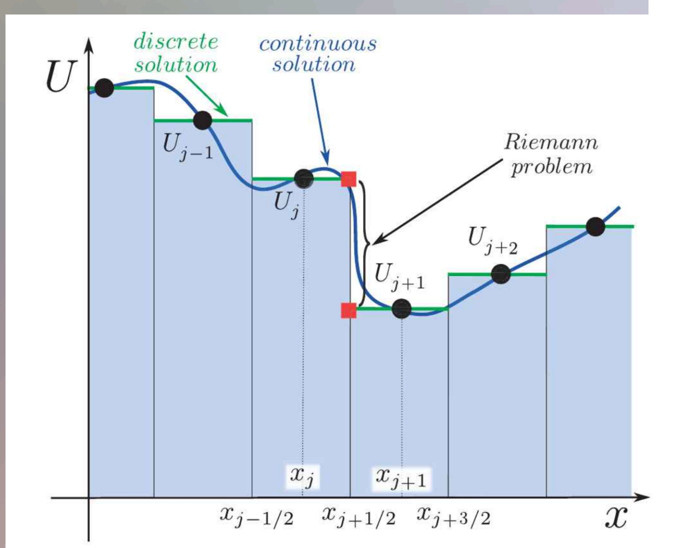
$$\nabla_\mu (\rho u^\mu) = 0\,,$$
$$\nabla_\nu T^{\mu\nu} = 0\,.$$

To guarantee that the numerical solution of the hydrodynamical equations (the conservation of rest mass and energy-momentum) converge to the right solution, they need to be reformulated into a conservative formulation. Most of the numerical “hydro codes” use here the 3+1 Valencia formulation.

# Finite difference methods

Discretisation of a hyperbolic initial value boundary problem.




High resolution shock capturing methods (HRSC methods) are needed, when Riemann problems of discontinuous properties and shocks needs to be evolved accurately

All figures from: Luciano Rezzolla, Olindo Zanotti: Relativistic Hydrodynamics, Oxford Univ. Press, Oxford (2013)

# The Einstein Toolkit

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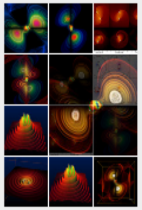
## Welcome



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The Einstein Toolkit Consortium is developing and supporting open software for relativistic astrophysics. Our aim is to provide the core computational tools that can enable new science, broaden our community, facilitate interdisciplinary research and take advantage of emerging petascale computers and advanced cyberinfrastructure.

Please read our pages about the Einstein Toolkit, its governance, and how to get started with the toolkit for more information.

### Download

**November 2014**: We are pleased to announce the tenth release (code name "Herschel") of the Einstein Toolkit, an open, community developed software infrastructure for relativistic astrophysics.

https://www.youtube.com/watch?v=EO4d32ch6OI
https://www.youtube.com/watch?v=p5bq2iUO3DE
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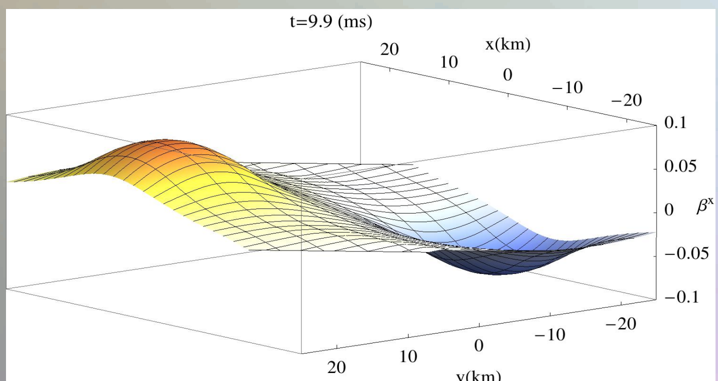
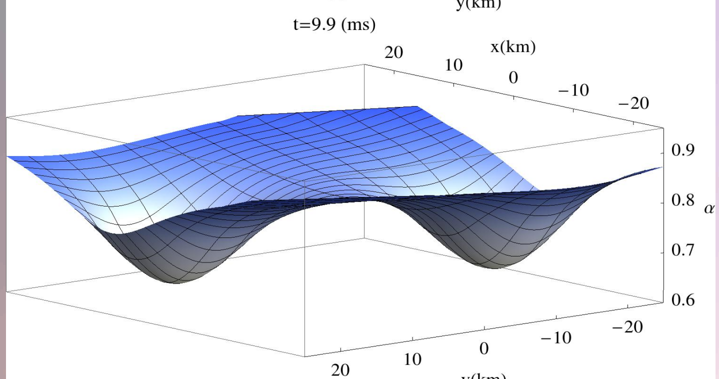
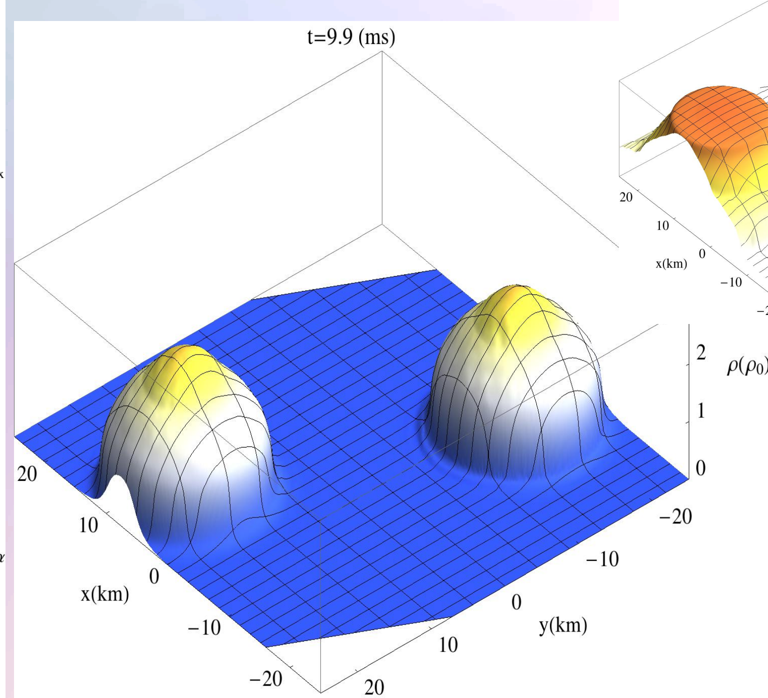
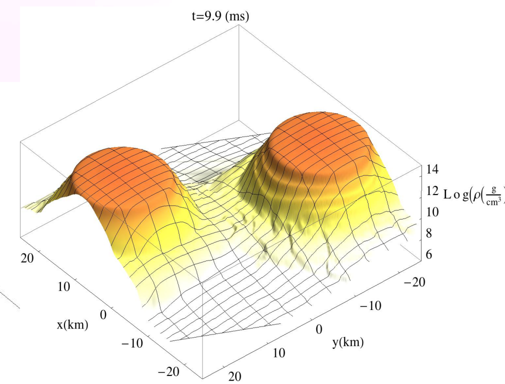
# Mergers of two Hybrid Stars

The frequency spectrum of the emitted gravitational wave reflects some of the properties of the equation of state (K.Takami, L.Rezzolla, and L.Baiott, "Constraining the Equation of State of Neutron Stars from Binary Mergers", arXiv:1403.5672). Whether a Hadron-Quark phase transition is present during merger should be visible using gravitational wave detectors. The following results represent a hybrid star merger calculation (EOS: ALF2 (APR-Hadronic model +Gibbs construction of MIT-Bag+CFL-Phase), see Alford et.al.,APJ 629,2005) where the EOS was fitted with a piecewise polytrope.



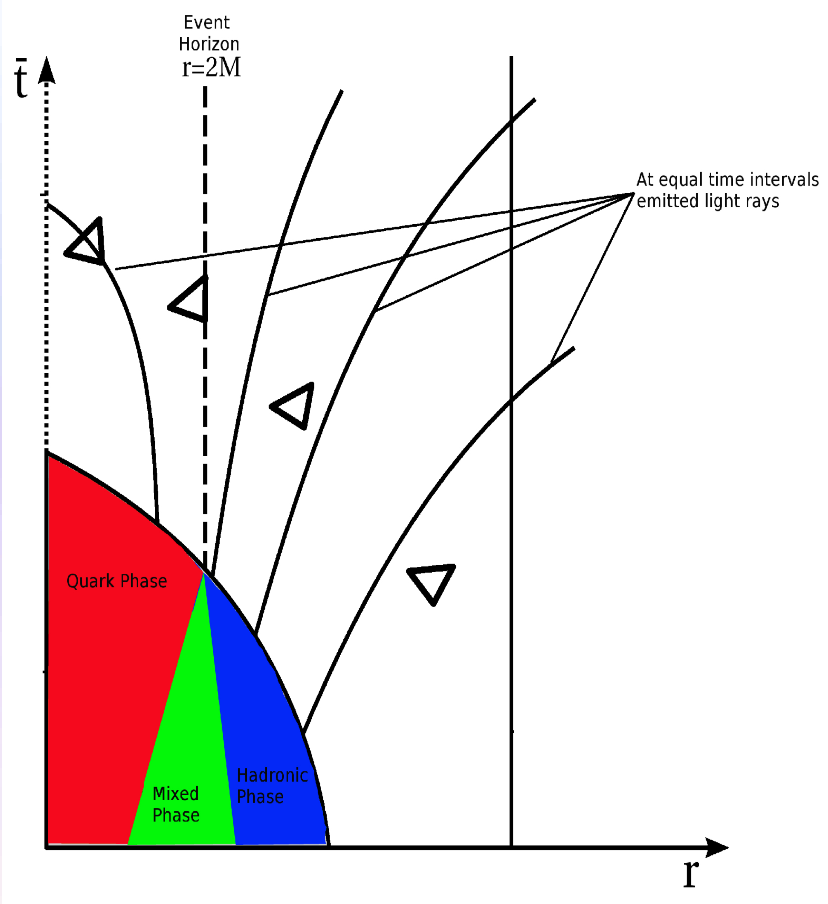




Simulations done by Kentaro Takami

# Collapse Scenario of a Hybrid Star

The gravitational collapse of a hybrid star to a black hole is visualized on the right side within a space-time diagram of the Schwarzschild metric in advanced Eddington-Finkelstein coordinates.

The formation of the apparent and event horizon of the black hole confines the quark star macroscopically. Finally the colour charge of the deconfined free quarks cannot be observed from outside.




M. Hanauske, Dissertation: "Properties of Compact Stars within QCD-motivated Models" ;
University Library Publication 2004 (urn:nbn:de:hebis:30-0000005872)

# Neutron-star oscillation