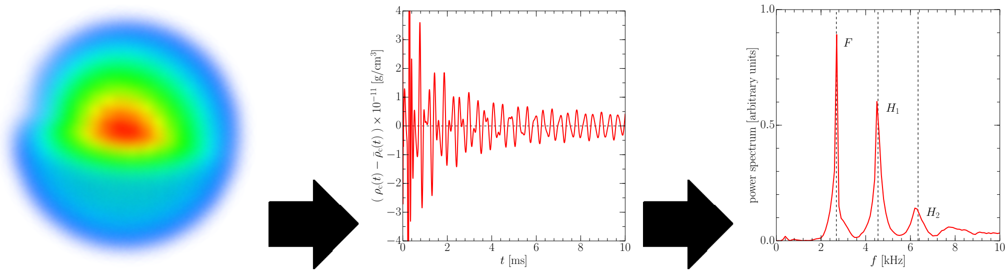

Computer Experiment :
Radial Oscillations of Relativistic Spherical Star

Tutor: Matthias Hanauske (e-mail:hanauske@th.physik.uni-frankfurt.de)

The setup and the description of this experiment was done by *Prof. Dr. Kentaro Takami*. He is now teaching at the 'Kobe City College of Technology' in Japan.

0 Class Information

- application form :
  - if you want to do this experiment, please register via e-mail to Matthias Hanauske no later than on the last Wednesday before the week in which you want to do the experiment; your e-mail should include the following information: (1) student number, (2) full name, (3) e-mail address.
- intensive course :
  - 12 [hours] = 6 [hours/week] × 2 [weeks].
- when :
  - Monday 9-16, two subsequent weeks upon individual arrangement with M. Hanauske; other time slots may be arranged with M. Hanauske individually.
- where :
  - Only Online (Zoom Meetings).
- preparation :
  - No preparatory work necessary.
- required skill :
  - basic Linux knowledge.
- using software :
  - `Einstein Toolkit` [1].
  - `scidata` (http://bitbucket.org/dradice/scidata).
  - `python` (https://www.python.org/) and `matplotlib` (http://matplotlib.org/).
  - `Mathematica`

1 Introduction

Stellar Oscillations give us many information of a star, such as an internal structure. Therefore they are well studied from the both sides of theory and observation, especially for the Sun as a *"helioseismology"*. These studies basically can be adapted to all of Newtonian stars as a *"asteroseismology"*. In fact, the same treatment are used for the pulsations of e.g., the Cepheid variable, and the properties are accurately extracted.

Basically we can use same idea for relativistic stars such as a neutron star. However the basic equations are very complicate and it is very difficult to analytically consider them. In fact, we do not know any analytic treatment for a rotating full relativistic star. Thanks to recent development of a numerical relativity and computer fluid dynamics, fortunately we can tackle the problems by using computer simulations. Then e.g., Ref. [3, 4, 9, 7] have been obtained.

In this experiment class, we learn how to extract oscillations of a relativistic star via computer simulations. The flowchart is (i) a construction of a star and the evolution in a computer, (ii) extraction of oscillations, and (iii) identification of the eigenfrequencies (see figures in top of page 1). We concentrate only radial oscillations of a spherical relativistic star, i.e., Tolman-Oppenheimer-Volkoff (TOV) solution, although these can be analytically computed by using a linear perturbation theory [8].

Unless explicitly stated, we use units in which $c = G = M_{\odot} = 1$ in this document.

2 Simulation and Analysis

2.1 Initial Data of TOV Star

Einstein's theory of general relativity in connection with the conservation laws for energy-momentum and rest mass are the groundings of the differential equations which we need to solve. The Einstein equation and the conservation laws are summarized in the following set of highly non-linear differential equations:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \;=\; 8\pi T_{\mu\nu} \;, \quad \nabla_{\mu} T^{\mu\nu} \;=\; 0 \;, \quad \nabla_{\mu} \left(\rho\, u^{\mu}\right) \;=\; 0 \quad , \tag{1}$$

where $T_{\mu\nu}$ is the energy-momentum tensor, $R_{\mu\nu}$ is the Ricci tensor, which contains first and second derivatives of the space-time metric $g_{\mu\nu}$, ∇_{μ} is the covariant derivative and u^{μ} is the four velocity of the star's fluid. The Einstein equation (first equation in (1)) describes in which way the space-time structure need to bend (left hand side of the equation) if energy-momentum is present (right hand side of the equation). In this experiment the energy-momentum, which curves space-time, arises from the large energy amount of the neutron star matter.

In the following we will focus on the solution of a static spherical relativistic star, which we will use as the initial data in our simulation code. In a static, spherical symmetrical condition, the metric $g_{\mu\nu}$ and the infinitesimal line element ds can be written as

$$ds^2 \;=\; g_{\mu\nu}\, dx^{\mu} dx^{\nu} \;=\; -\, e^{2\nu(r)} dt^2 + \left(1 - \frac{2m(r)}{r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \; d\phi^2 \quad . \tag{2}$$

In the following we will use the expression of an ideal fluid energy-momentum tensor to describe the neutron star matter

$$T_{\mu\nu} \;=\; (e + p)\, u_{\mu} u_{\nu} + p\; g_{\mu\nu} \;, \quad e \;=\; \rho\,(1 + \epsilon) \quad , \tag{3}$$

where e is the energy density, p the pressure, ρ the rest mass density and ϵ the internal energy density of the neutron star fluid.

By inserting the Ansatz of the metric (2) together with the expression for the energy momentum tensor (3) into the equations (1) one derives the Tolman-Oppenheimer-Volkoff (TOV) equations (see e.g., [5]):

$$\frac{dm}{dr} = 4\pi r^2 \, e \ , \quad \frac{d\nu}{dr} = \frac{m + 4\pi r^3 p}{r\,(2m - r)} \ , \quad \frac{dp}{dr} = -\,(e+p)\frac{d\nu}{dr} \quad . \tag{4}$$

For a given equation of state (EOS: a function $p(e)$ or $p(\rho)$) the TOV equation can be solved numerically by fixing the central pressure p_c and integrating outwards to the star's surface where the pressure is zero. In our simulation we will use a simple polytropic EOS: $p = K\rho^\Gamma$, $\Gamma = 2$ and $K = 100$, with $e = \rho + p/(\Gamma - 1)$.

2.2 Time Evolution of Initial Data

The TOV star is perturbed by simply a numerical noise in this experiment. Then the perturbed star is numerically evolved by solving the relativistic hydro equations under (a) the fixed spacetime (Cowling approximation), or (b) the dynamical spacetime in Baumgarte-Shapiro-Shibata-Nakamura-Oohara-Kojima (BSSNOK) formulation [6, 2] which is modification of the 3 + 1 ADM formalism. In practice, for these purposes we use a publicly available `Einstein Toolkit` [1].

2.3 Extraction of Radial Oscillations

We extract the radial oscillations from the time variation of the central density $\rho_c(t)$ as in a figure (ii) in top of page 1. Then the frequencies of a fundamental (F) and the overtone modes ($H_1, H_2, \cdots$) for the radial oscillations are identified via a discrete Fourier transformation (DFT) (see figure (iii) in top of page 1).

3 Tasks

3.1 Task 1: Find out more about the Einstein Toolkit

In the experiment we will use the simulation results of the publicly available `Einstein Toolkit` [1] computer code. Find out about the general structure of this computer program and take a special look at the following values defined in the parameter file (.par file):

- Activation of thorns.
- Definition of the initial data.
- Setup of the numerical grid.
- Definition of the equation of state.
- Definition of the space-time evolution.
- Definition of the output files.

3.2 Task 2: Simulation of Oscillated TOV Star

Analyse the simulated data of the time evolution of an oscillated TOV star with $\rho_c(t = 0) = 0.00128$ in Cowling approximation by a low grid resolution $dx = dy = dz = 0.4$:

- understand the units of the code, i.e., $c = G = M_\odot = 1$, and the relation to CGS units.

- understand the model and the parameter file.
- understand the outputted data files, and plot the data by using e.g., `python` or `mathematica`.

3.3 Task 3: Identification of Eigenfrequencies for Radial Oscillations

Compute Power spectral density (PSD) of the variation of the central density $\rho(t)$, and then compare the frequencies with the exact eigenfrequencies ($F \approx 2.686$[kHz], $H_1 \approx 4.550$[kHz], $H_2 \approx 6.342$[kHz], $H_3 \approx 8.108$[kHz]) computed by a linear perturbation theory:

- extract the variation of the central density $\rho(t)$ from simulation data.
- compute the PSD and check your peak frequencies by using the exact eigenfrequencies.
- prepare a figure such as a figure (iii) in top of page 1.

3.4 Task 4: Dependency for Grid Resolutions

In above tasks, you had a simulation with low grid resolution $dx = dy = dz = 0.4$, and extracted the frequencies. Of course the results has numerical error proportion to the grid resolution. Therefore, analyse the data of a middle ($dx = dy = dz = 0.3$) and high ($dx = dy = dz = 0.2$) grid resolution:

- please do same things with low grid resolution case for middle and high resolution cases.
- compare the results for three resolutions (low, middle, high) and make clear the differences. For example, please plot $\rho_c(t)$ for three resolutions.

3.5 Task 5: Oscillated TOV Star in Full Spacetime Evolution

In above, you assumed the Cowling approximation in the simplicity and cheap numerical cost. In order to check whether the approximation is appropriate or not, please analyse the data of the same model without the Cowling approximation, i.e., in full general relativistic evolution with $dx = dy = dz = 0.4$:

- show the difference to the Cowling approximation by e.g., plotting the time evolution of the lapse function.
- compute and extract the frequencies.
- compare the frequencies to the case with the Cowling approximation. Is there difference? Is the approximation appropriate for the radial oscillations?

4 Final Report

You have to submit the report within 4 weeks after performing the simulation:

- please use the units such as kHz, g/cm^3, ms in the report, although $c = G = M_\odot = 1$ units is used in the code.
- please summarize what you learn and the results you get in this experiment.
- please include figures you plotted in the tasks.
- please tell me the frequencies which you identified in Task 5. At least, F and H_1 frequencies can be extracted.

References

[1] `http://einsteintoolkit.org`.

[2] T. W. Baumgarte and S. L. Shapiro. Numerical integration of Einstein's field equations. *Phys. Rev. D*, 59(2):024007, January 1999.

[3] H. Dimmelmeier, N. Stergioulas, and J. A. Font. Non-linear axisymmetric pulsations of rotating relativistic stars in the conformal flatness approximation. *Mon. Not. R. Astron. Soc.*, 368:1609–1630, June 2006.

[4] E. Gaertig and K. D. Kokkotas. Oscillations of rapidly rotating relativistic stars. *Phys. Rev. D*, 78(6):064063, sep 2008.

[5] L. Rezzolla and O. Zanotti. *Relativistic Hydrodynamics*. Oxford University Press, Oxford, UK, 2013.

[6] M. Shibata and T. Nakamura. Evolution of three-dimensional gravitational waves: Harmonic slicing case. *Phys. Rev. D*, 52:5428–5444, November 1995.

[7] K. Takami, L. Rezzolla, and S. Yoshida. A quasi-radial stability criterion for rotating relativistic stars. *Mon. Not. R. Astron. Soc.*, 416:L1–L5, September 2011.

[8] S. Yoshida and Y. Eriguchi. Quasi-radial modes of rotating stars in general relativity. *Mon. Not. R. Astron. Soc.*, 322:389, 2001.

[9] B. Zink, O. Korobkin, E. Schnetter, and N. Stergioulas. Frequency band of the f-mode Chandrasekhar-Friedman-Schutz instability. *Phys. Rev. D*, 81(8):084055, April 2010.