

Vorlesung 4

Allgemeine Relativitätstheorie mit dem Computer

PC-Pool Raum 01.120 Johann Wolfgang Goethe Universität 02. Mai, 2016

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Allgemeines

Ort und Zeit:

PC-Pool Raum 01.120, immer Montags von 16.15 bis 17.45 Uhr
Zusätzlicher, freiwilliger Übungstermin 15.00 bis 16.15 Uhr

Vorlesungs-Materialien und *Lon Capa* Online-Lernplattform:

<http://th.physik.uni-frankfurt.de/~harauske/VARTC/>

<http://lon-capa.server.uni-frankfurt.de/>

Plan für die heutige Vorlesung:

Wiederholung: Geodätengleichung in Schwarzschild-Raumzeit für unterschiedliche Anfangsbedingungen mit Maple lösen, Klassifizierung unterschiedlicher Bahnbewegungen mittels eines effektiven Potentials

Animationen und Schleifen in Maple

Die innerste stabile kreisförmige Bahnbewegung (ISCO) eines Probekörpers
Bewegung von Licht um ein schwarzes Loch, die Photonensphäre

Einführung: Weiße Zwerge, Neutronensterne, Hybride Sterne und Quarksterne

Die Geodätengleichung

Maple Output für die Schwarzschildmetrik:

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma_{\nu\rho}^\mu \frac{dx^\nu}{d\tau} \frac{dx^\rho}{d\tau} = 0$$

$$\begin{aligned} \text{eqns} := & \left\{ \frac{d^2}{d\lambda^2} \tau(\lambda) - \frac{2M \left(\frac{d}{d\lambda} \tau(\lambda) \right) \left(\frac{d}{d\lambda} r(\lambda) \right)}{r(-r+2M)} = 0, \frac{d^2}{d\lambda^2} \phi(\lambda) \right. \\ & + \frac{2 \left(\frac{d}{d\lambda} r(\lambda) \right) \left(\frac{d}{d\lambda} \phi(\lambda) \right)}{r} + \frac{2 \cos(\theta) \left(\frac{d}{d\lambda} \theta(\lambda) \right) \left(\frac{d}{d\lambda} \phi(\lambda) \right)}{\sin(\theta)} = 0, \frac{d^2}{d\lambda^2} \theta(\lambda) \\ & + \frac{2 \left(\frac{d}{d\lambda} r(\lambda) \right) \left(\frac{d}{d\lambda} \theta(\lambda) \right)}{r} - \sin(\theta) \cos(\theta) \left(\frac{d}{d\lambda} \phi(\lambda) \right)^2 = 0, \frac{d^2}{d\lambda^2} r(\lambda) \\ & - \frac{(-r+2M)M \left(\frac{d}{d\lambda} \tau(\lambda) \right)^2}{r^3} + \frac{M \left(\frac{d}{d\lambda} r(\lambda) \right)^2}{r(-r+2M)} + (-r+2M) \left(\frac{d}{d\lambda} \theta(\lambda) \right)^2 + (\\ & \left. -r+2M) \sin(\theta)^2 \left(\frac{d}{d\lambda} \phi(\lambda) \right)^2 = 0 \right\} \end{aligned}$$

Anfangsbedingungen

Zum Lösen des Systems von Differentialgleichungen müssen die Anfangsbedingungen des Probekörpers festgelegt werden (z.B. wo befindet sich der Probekörper zur Eigenzeit $\lambda=0$: $r(0)=10$)

Die Anfangsbedingung für $t'(0)$ erhält man z.B. mittels der folgenden Bedingungen aus dem infinitesimalen Weglängenelement ds^2 :

$$\frac{ds^2}{d\lambda^2} = 1 \quad \underbrace{\Rightarrow}_{dr=d\theta=d\phi=0 \text{ bei } t=0} \quad \left(1 - \frac{2M}{r}\right) \frac{dt^2}{d\lambda^2} = 1 \quad \Rightarrow \quad \frac{dt}{d\lambda} = \frac{1}{\sqrt{\left(1 - \frac{2M}{r}\right)}}$$

$$\frac{ds^2}{d\lambda^2} = 1 \quad \underbrace{\Rightarrow}_{dr=d\theta=0, \theta=\pi/2 \text{ bei } t=0} \quad \left(1 - \frac{2M}{r}\right) \frac{dt^2}{d\lambda^2} - r^2 \frac{d\phi^2}{d\lambda^2} = 1 \quad \Rightarrow \quad \frac{dt}{d\lambda} = \sqrt{\frac{1 + r^2 \frac{d\phi^2}{d\lambda^2}}{\left(1 - \frac{2M}{r}\right)}}$$

Während der Bewegung erhaltenen Größen

Zwei Gleichungen der Geodätengleichung lassen sich wie folgt umformulieren:

$$\begin{aligned} 1. \text{ Gleichung: } & \frac{d}{d\lambda} \left[\left(1 - \frac{2M}{r} \right) \frac{dt}{d\lambda} \right] = 0 \quad \rightarrow \quad \left(1 - \frac{2M}{r} \right) \frac{dt}{d\lambda} = E = \text{const} \\ 2. \text{ Gleichung: } & \frac{d}{d\lambda} \left(r^2 \sin^2(\theta) \frac{d\phi}{d\lambda} \right) = 0 \quad \rightarrow \quad r^2 \sin^2(\theta) \frac{d\phi}{d\lambda} = l = \text{const} \quad , \end{aligned}$$

Wobei die während der Bewegung erhaltenen Größen (E und l) sich mittels des Viererimpulses definieren:

Viererimpulses $p_\mu = m u_\mu$ ergeben ($\lambda = \tau$):

$$\begin{aligned} p_0 &= m \frac{dx_0}{d\tau} = m g_{00} \frac{dx^0}{d\tau} = m g_{00} \frac{dt}{d\tau} = m \left(1 - \frac{2M}{r} \right) \frac{dt}{d\tau} = m E \\ p_3 &= m \frac{dx_3}{d\tau} = m g_{33} \frac{dx^3}{d\tau} = m g_{33} \frac{d\phi}{d\tau} = -m \left(r^2 \sin^2(\theta) \right) \frac{d\phi}{d\tau} = -m l \end{aligned}$$

Das effektive Potential

Mittels der radialen Gleichung läßt sich das effektive Potential def.:

$$4. \text{ Gleichung: } \rightarrow \frac{1}{2} \left(\frac{dr}{d\tau} \right)^2 + V(r, M, l) = \frac{1}{2} (E^2 - 1)$$

$$\text{wobei: } V(r, M, l) = \frac{l^2}{2r^2} \left(1 - \frac{2M}{r} \right) - \frac{M}{r} ,$$

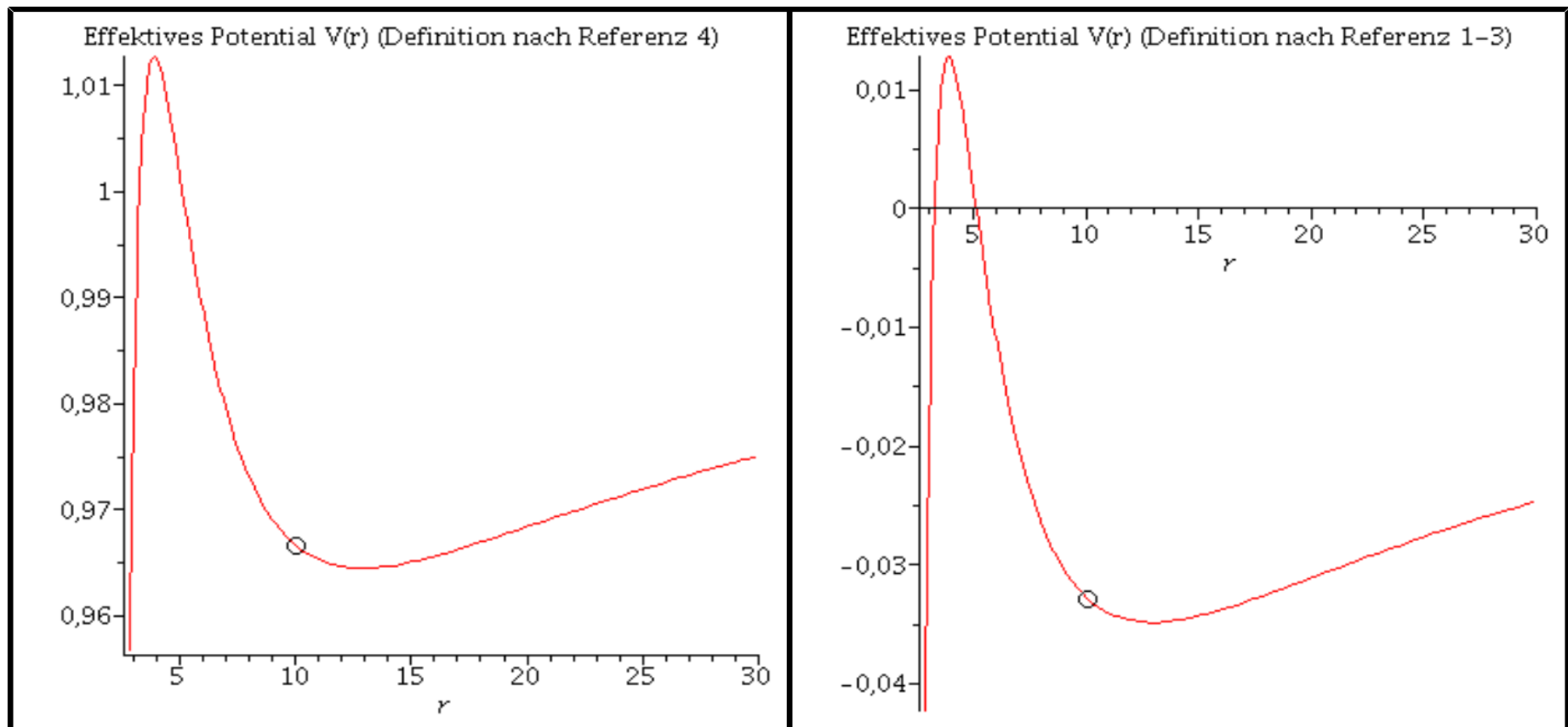
Alternative Definition (siehe Buch von Prof. Rezzolla):

$$4. \text{ Gleichung: } \rightarrow \left(\frac{dr}{d\tau} \right)^2 + (V(r, M, l))^2 = E^2$$

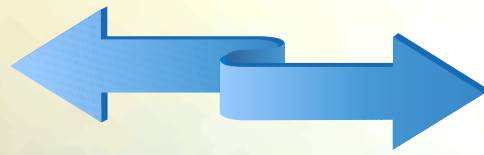
$$\text{wobei: } V(r, M, l) = \sqrt{\left(1 - \frac{2M}{r} \right) \left(1 + \frac{l^2}{r^2} \right)} ,$$

Das effektive Potential

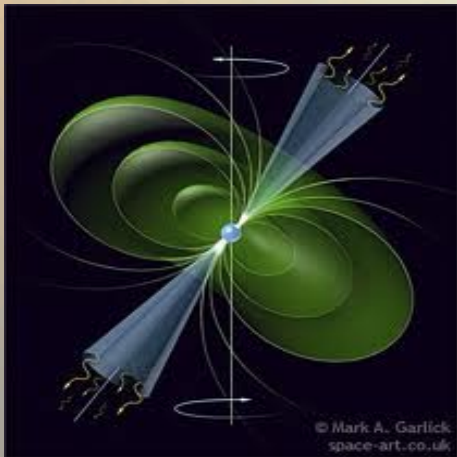
Beispiel



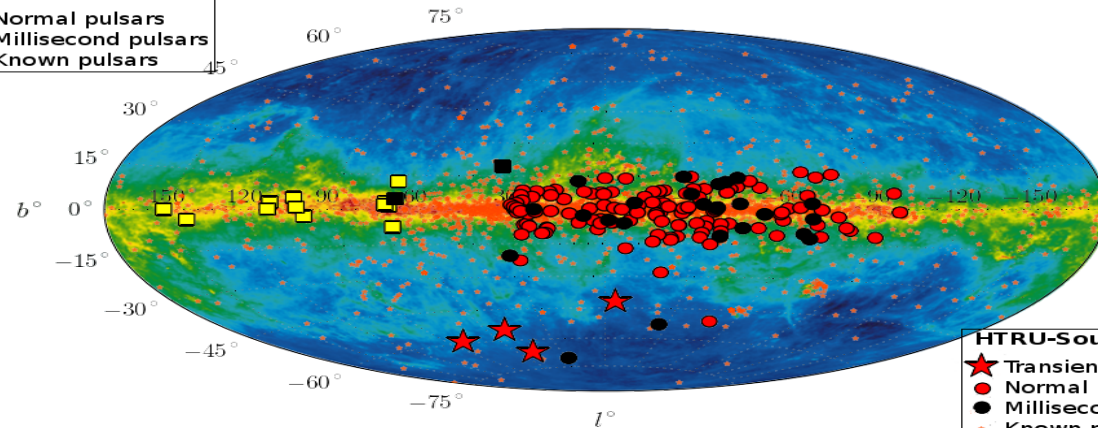
Pulsars



Neutron Stars

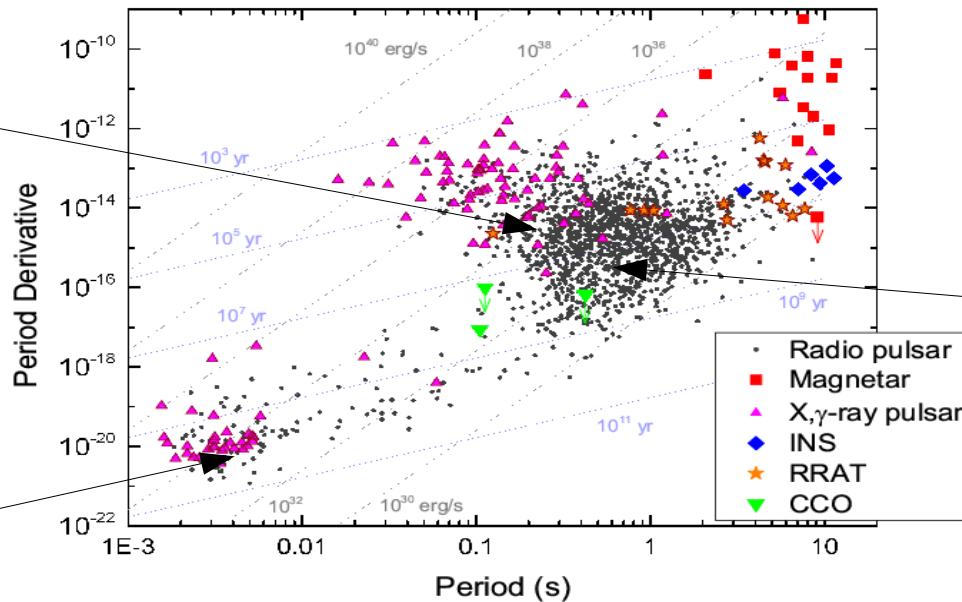


HTRU-North:
 ■ Normal pulsars
 ■ Millisecond pulsars
 * Known pulsars



HTRU-South:
 ★ Transient bursts
 ● Normal pulsars
 ● Millisecond pulsars
 * Known pulsars

PSR B0531+21 (33.5 ms)
Crab Pulsar



PSR B0329+54 (0.715 s)



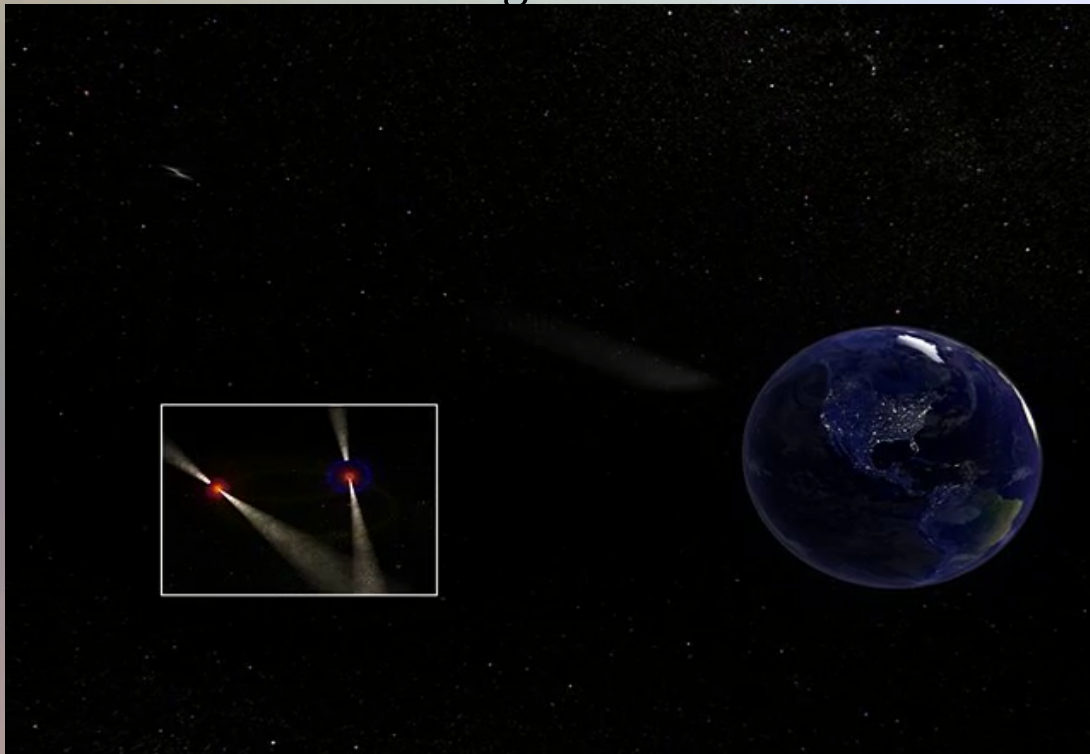
PSR B1937+21 (1.56 ms)

Binary Neutron Stars

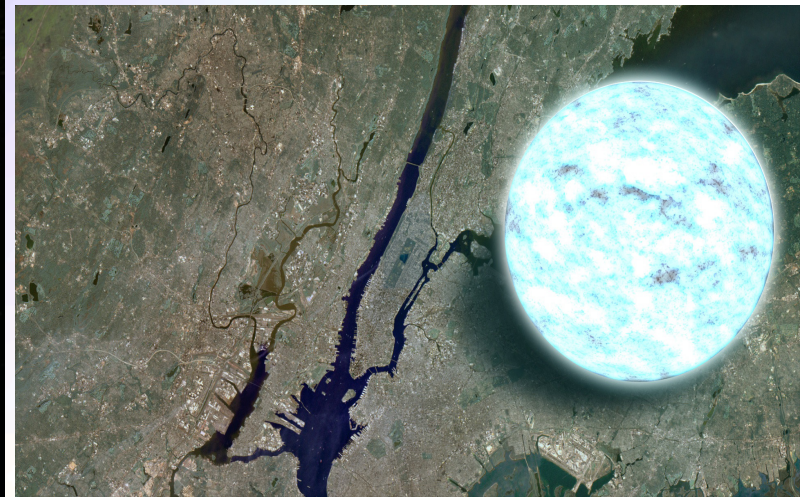
~ 2500 neutron stars are known, large magnetic fields (up to 10^{11} Tesla), fast rotation (up to 700 rotations/second), radius ~ 10 km, mass 1-2 solar masses.

Some NS are in binary systems (NS-planet, NS-(white dwarf) or NS-NS).

Double Pulsar (PSR J0737-3039A/B), discovered in 2003, separated only by 800,000 km, orbital period of 147 minutes, Periodic eclipse of one pulsar by the other, emission of gravitational waves \rightarrow will merge in 85 million years.



Size of a neutron star compared to
New York



McGill NCS Multimedia Services Animation by Daniel Cantin, DarwinDimensions)

NASA/Goddard Space Flight Center

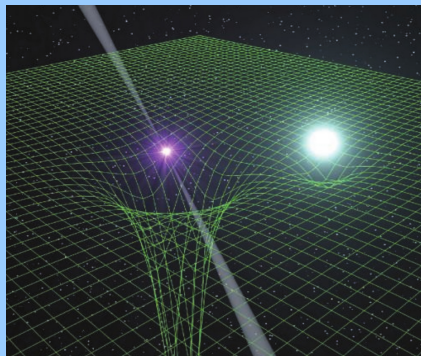
Observed Masses of Compact Star Binaries

PSR J1906+0746 Van Leeuwen et al, arXiv:1411.1518

144-ms pulsar, discovered in in 2004
 Orbital period: 3.98 hours, Eccentricity: 0.085
 Pulsar mass: 1.291(11), Companion mass 1.322(11)
 Observed between 1998-2009,
 then it disappeared due to spin precession

Double Pulsar PSR J0737-3039

Orbital period: 147 min, Eccentricity: 0.088
 pulsar A: P=23 ms, M=1.3381(7)
 pulsar B: P=2.7 s, M=1.2489(7)
 Pulsar A is eclipsed once per orbit by B (for 30 s)
 Kramer, Wex, Class. Quantum Grav. 2009



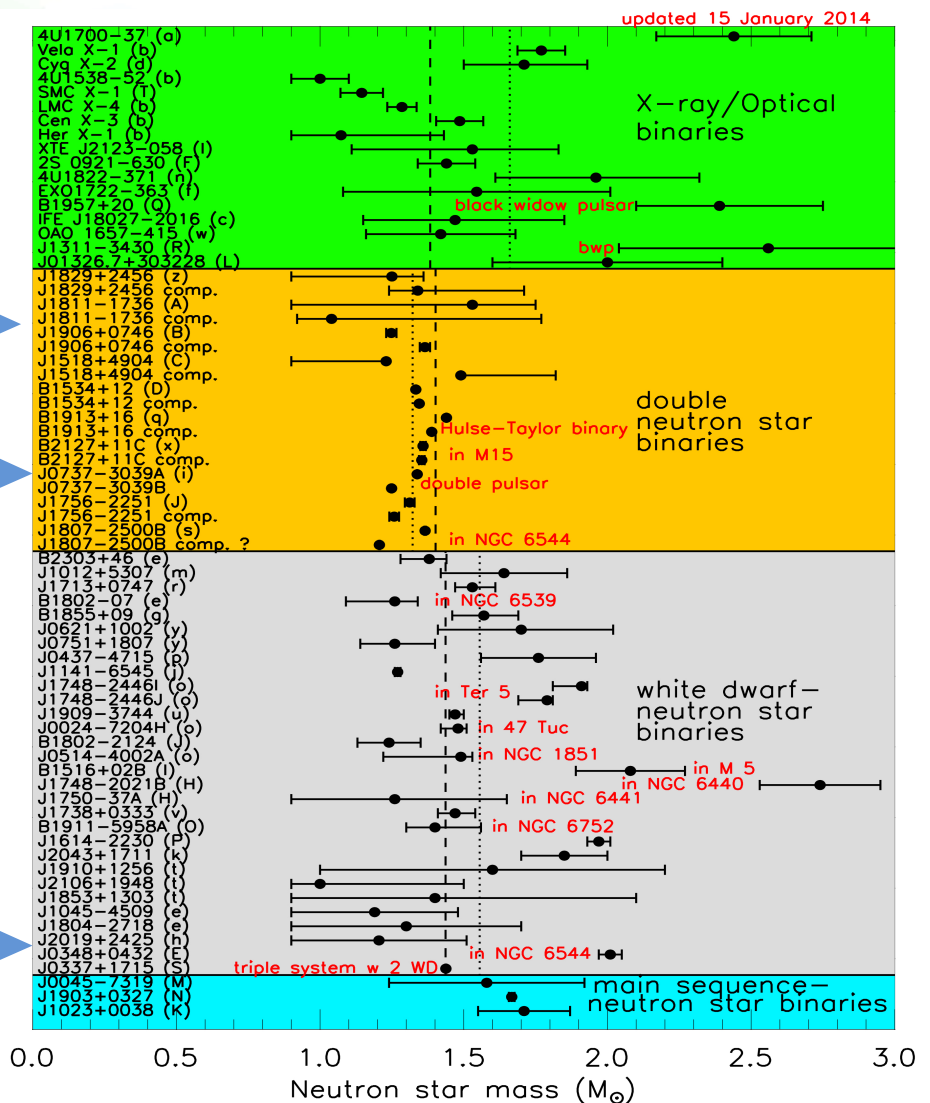
Picture from J. Antoniadis et.al. Science 2013

PSR J0348+0432

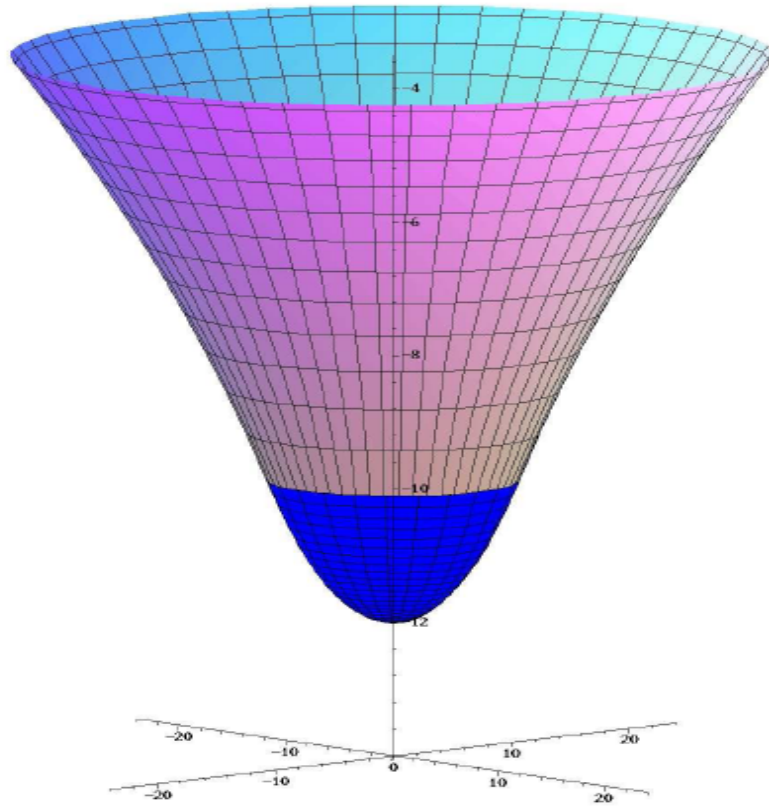
Orbital Period:
2.46 hours

Pulsar mass:
2.01±0.04

white dwarf mass:
0.172±0.003



Neutron Stars



Black Holes

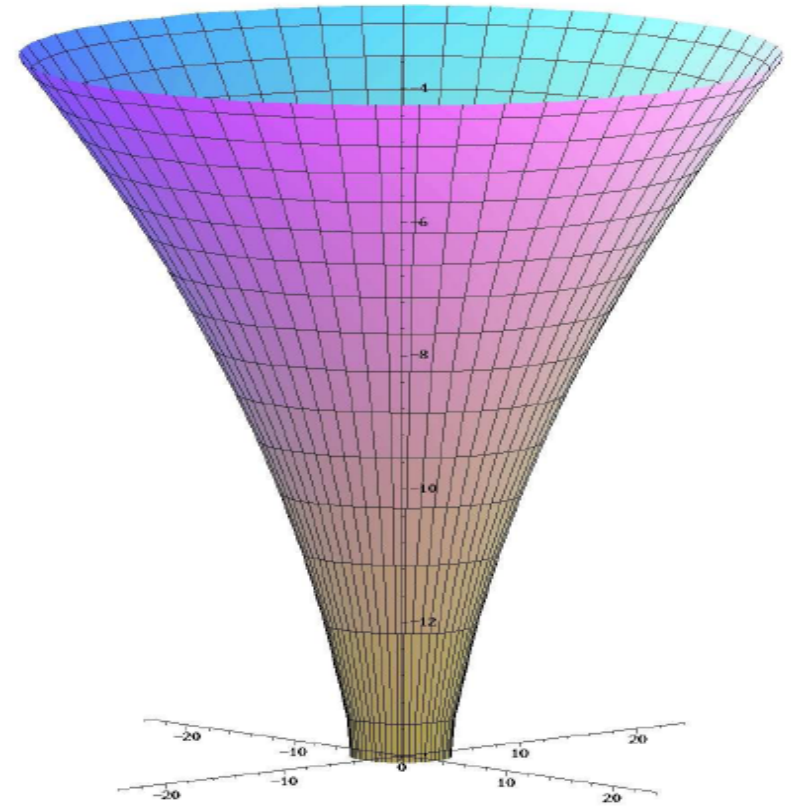


Abbildung 2.5: Eingebettetes Raumzeitdiagramm eines Neutronensterns (links) und eines schwarzen Loches (rechts) wobei $M = 1.4 M_{\odot}$ und die x- und y-Achse in Einheiten km dargestellt sind.

The Einstein Equation

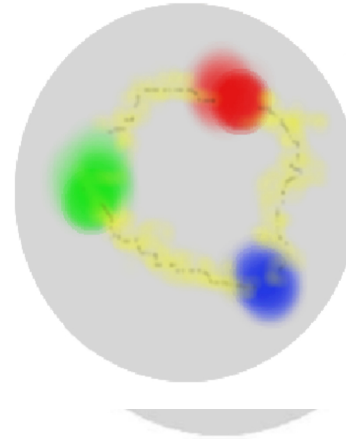
ART	Yang-Mills-Theories
$D_\beta v^\alpha = \partial_\beta v^\alpha + \Gamma_{\sigma\beta}^\alpha v^\sigma$	$D_{\beta a}{}^b = \partial_\beta 1_a{}^b + ig A_{\beta a}{}^b$
$R^\delta{}_{\mu\alpha\beta} v^\mu = [D_\alpha, D_\beta] v^\delta$	$F_{\alpha\beta a}{}^b = \frac{1}{ig} [D_{\alpha a}{}^c, D_{\beta c}{}^b]$
$R^\delta{}_{\mu\alpha\beta} = \Gamma_{\mu\alpha \beta}^\delta - \Gamma_{\mu\beta \alpha}^\delta$ $+ \Gamma_{\nu\beta}^\delta \Gamma_{\mu\alpha}^\nu + \Gamma_{\nu\alpha}^\delta \Gamma_{\mu\beta}^\nu$	$= A_{\beta a}{}^b{}_{ \alpha} - A_{\alpha a}{}^b{}_{ \beta}$ $+ \frac{1}{ig} [A_{\alpha a}{}^c, A_{\beta c}{}^b]$
$\mathcal{L}_G = R + \underbrace{(c_1 R_{\mu\nu} R^{\mu\nu} + \dots)}_{\equiv 0 \text{ for ART}}$	$\mathcal{L}_{YM} = \frac{1}{4} F_{\mu\nu a}{}^b F^{\mu\nu}{}_a{}^b$

Quantum ChromoDynamic:

($SU(3)_{(c)}$ - Color Yang-Mills-Gauge Theory)

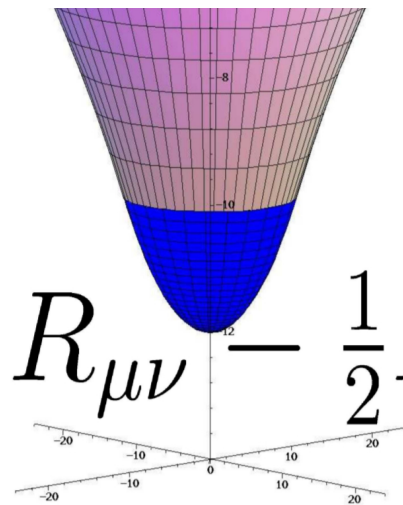
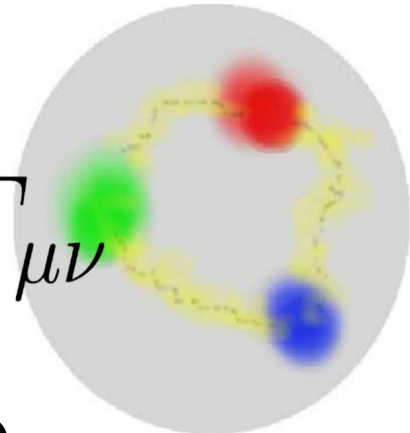
$$D_{\beta A}{}^B = \partial_\beta 1_A{}^B + ig G_{\beta A}{}^B$$

$A, B = \text{red, green, blue}$



$$\psi_A^f = \begin{pmatrix} \psi_r^f \\ \psi_g^f \\ \psi_b^f \end{pmatrix}$$

Confinement
chiral symmetry, ...

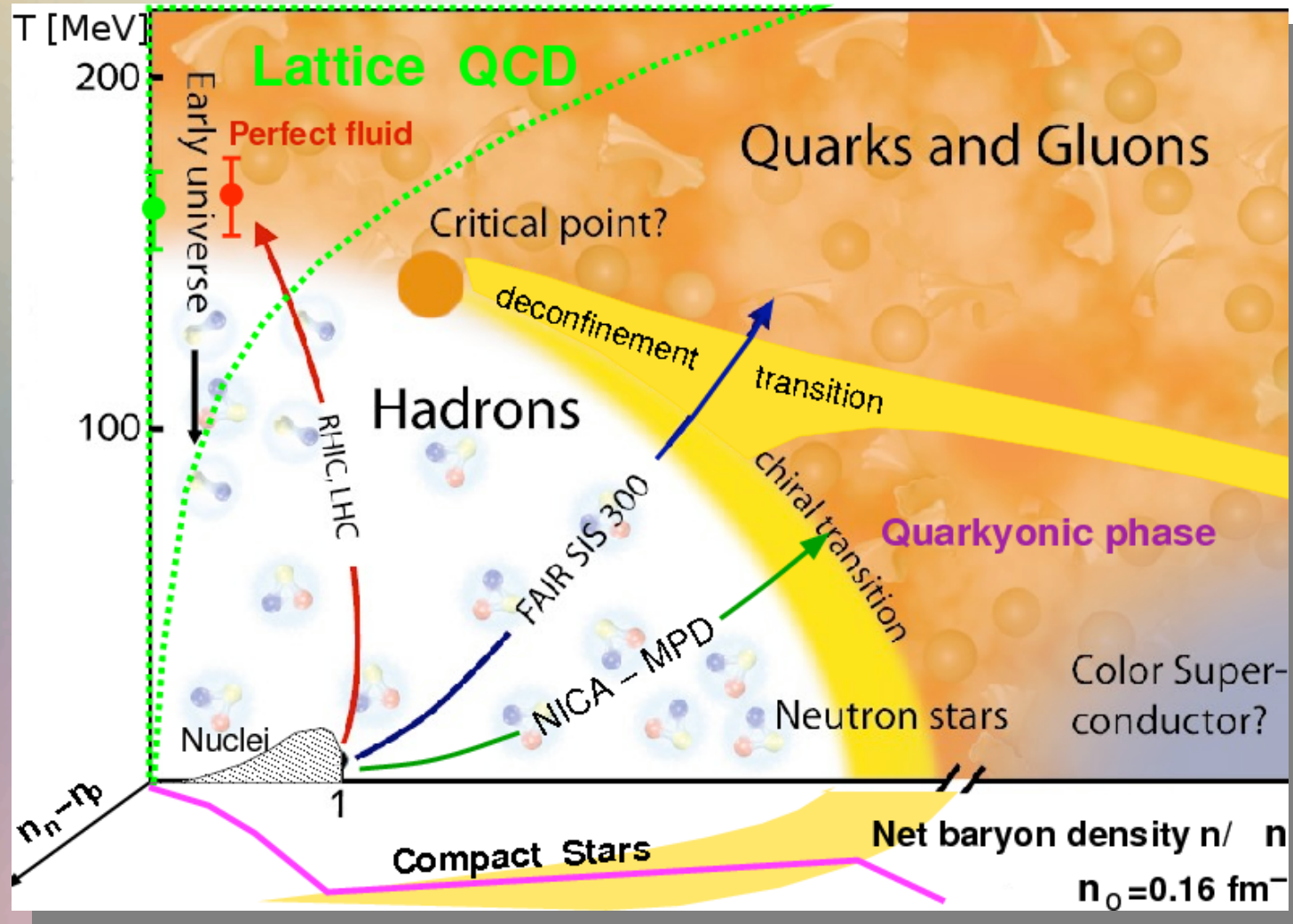


$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} =$$

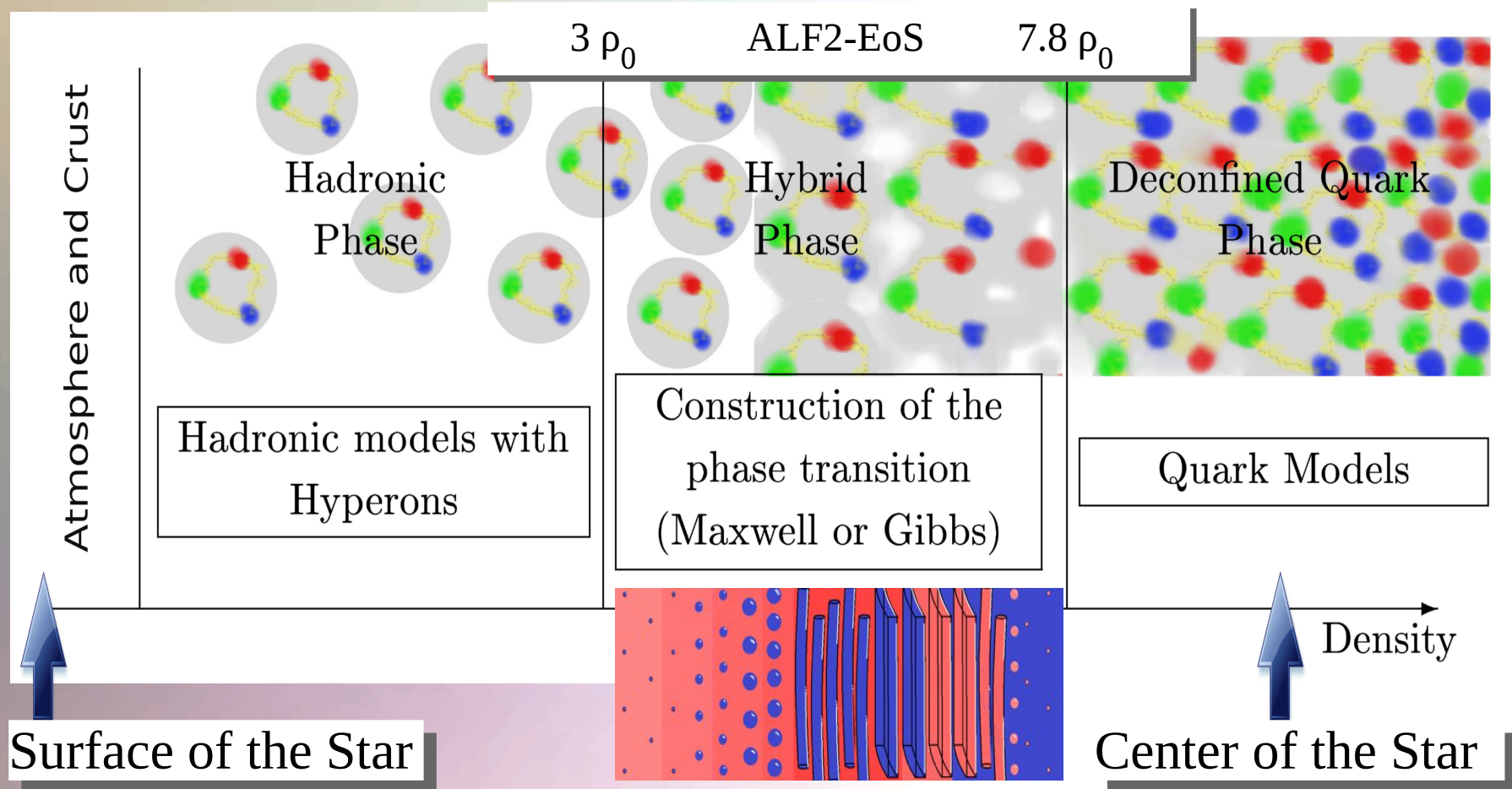
$$\frac{8\pi G}{c^4} T_{\mu\nu}$$

EOS: $P(\rho, T)$

The Equation of State and the QCD Phase Diagram



The QCD – Phase Transition and the Interior of a Hybrid Star

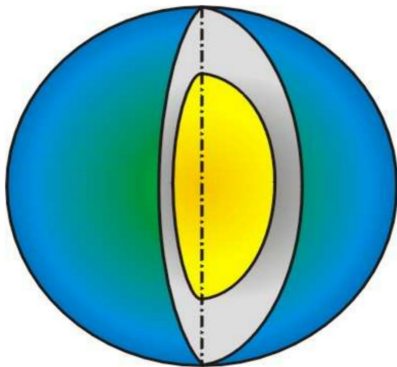


See: *Stable hybrid stars within a SU(3) Quark-Meson-Model*,
A.Zacchi, M.Hanuske, J.Schaffner-Bielich, PRD 93, 065011 (2016)

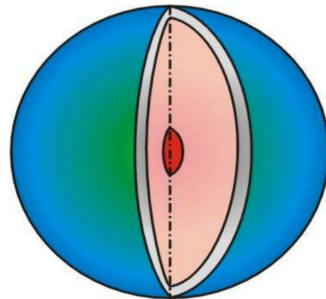
The Compact Star Zoo

Depending on the model used, the compact star zoo consists of different inhabitants: e.g. neutron stars with and without hyperons, quark stars and strange quark stars, hybrid stars with color superconducting quark matter, hybrid stars with Bose-Einstein condensates of antikaons.

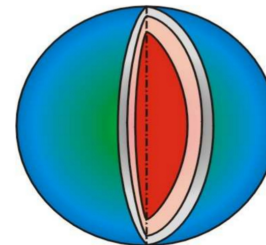
Neutron Stars



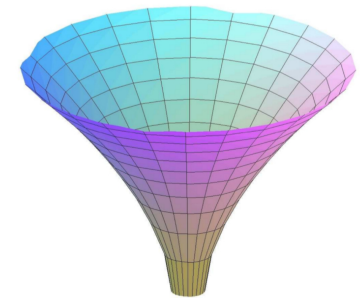
Hybrid Stars



Quark Stars



Black Holes



$$\rho_c = \rho_0$$

$$\approx 2 \rho_0$$

$$\approx 5 \rho_0$$

... ∞

Central density ρ_c in the star

$$(\rho_0 := 0.15/\text{fm}^3)$$

Neutron Stars

$$g_{\mu\nu} = \begin{pmatrix} e^{\nu(r)} & 0 & 0 & 0 \\ 0 & -e^{\lambda(r)} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2 \sin^2 \theta \end{pmatrix} . \quad (2.45)$$

Das Einsetzen dieses Ansatzes der Metrik in die Einsteingleichung

$$G^\mu{}_\nu = R^\mu{}_\nu - \frac{1}{2} R g^\mu{}_\nu = 8\pi\kappa T^\mu{}_\nu \quad (2.46)$$

liefert das folgende System von Differentialgleichungen:

$$\begin{aligned} G^t{}_t &= -e^{-\lambda} \left(\frac{1}{r^2} - \frac{\lambda'}{r} \right) + \frac{1}{r^2} &= 8\pi\kappa T^t{}_t \\ G^r{}_r &= -e^{-\lambda} \left(\frac{1}{r^2} + \frac{\nu'}{r} \right) + \frac{1}{r^2} &= 8\pi\kappa T^r{}_r \\ G^\theta{}_\theta &= -\frac{e^{-\lambda}}{2} \left(\nu'' - \frac{\lambda'\nu'}{2} + \frac{(\nu')^2}{2} + \frac{\nu' - \lambda'}{r} \right) &= 8\pi\kappa T^\theta{}_\theta \\ G^\phi{}_\phi &= G^\theta{}_\theta &= 8\pi\kappa T^\phi{}_\phi \end{aligned} \quad (2.47)$$

Neutron Stars

1..3, $i \neq j$) vernachlässigen. Der Energieimpulstensor $T^{\mu\nu}$ einer solchen idealen Flüssigkeit, lokal betrachtet an seinem Ort, kann wie folgt geschrieben werden

$$T^{\mu\nu} = (\epsilon + P)u^\mu u^\nu - g^{\mu\nu}P \quad \text{mit: } u^\mu = \frac{dx^\mu}{d\tau} \quad , \quad (2.48)$$

wobei u^μ die 4er Geschwindigkeit der Materie ist, τ die lokale Eigenzeit an einem betrachteten Materiepunkt beschreibt ($d\tau = \sqrt{ds^2} = \sqrt{g_{tt}} dt$, t ist die Koordinatenzeit eines unendlich entfernten Beobachters), ϵ die Energiedichte und P der Druck der Materie ist.

Neutron Stars

als die **Tollman-Oppenheimer-Volkoff (TOV) Gleichungen**

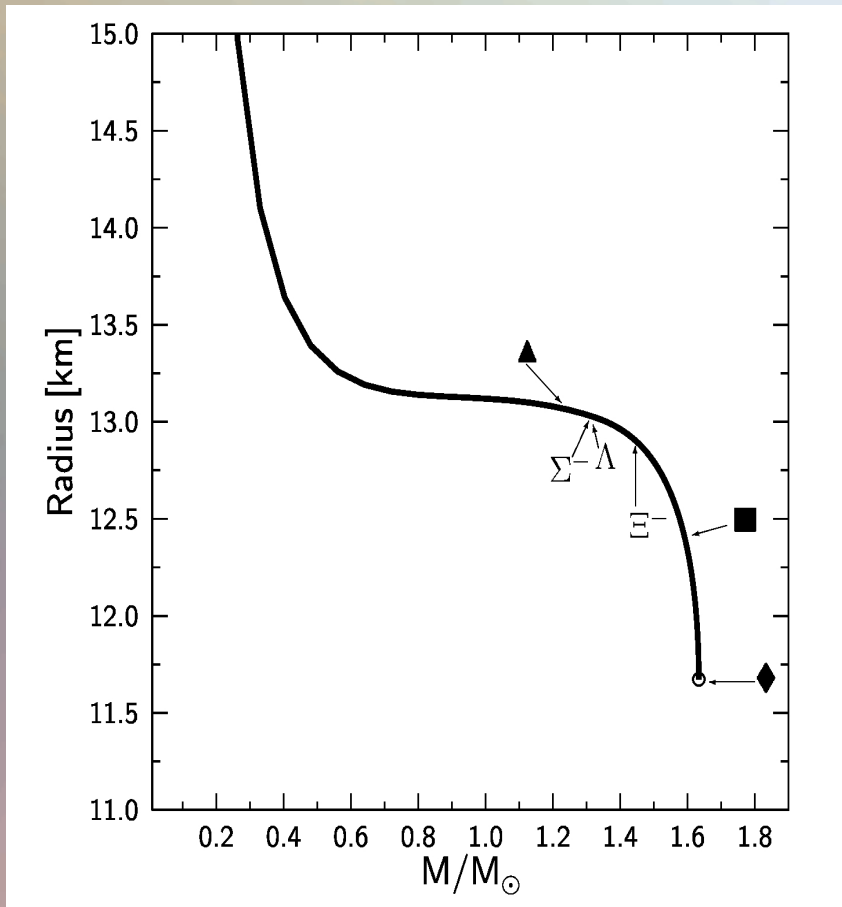
$$\begin{aligned}\frac{dP}{dr} &= -\frac{(\epsilon + P)4\pi r^3 + m}{r(r - 2m)} \\ m(r) &= \int_0^r 4\pi \tilde{r}^2 \epsilon(\tilde{r}) d\tilde{r} \\ \frac{d\nu}{dr} &= \frac{8\pi P r^3 + 2m}{r(r - 2m)},\end{aligned}\tag{2.61}$$

wobei die raumzeitliche Struktur durch die folgenden Ausdrücke bestimmt ist

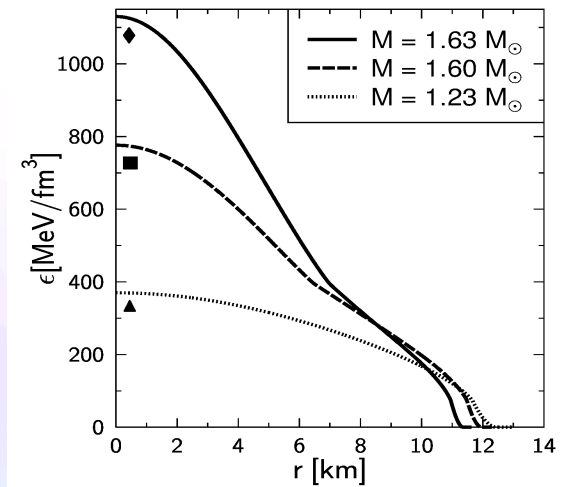
$$g_{\mu\nu} = \begin{pmatrix} e^{\nu(r)} & 0 & 0 & 0 \\ 0 & -\left(1 - \frac{2m(r)}{r}\right)^{-1} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2 \sin^2 \theta \end{pmatrix}$$
$$ds^2 = e^{\nu(r)} dt^2 + \left(1 - \frac{2m(r)}{r}\right)^{-1} dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2).$$

Neutron Star Properties

The neutron star radius as a function of its mass. A low, middle and high density star is displayed within the figure. Additionally the onset of hyperonic particles is visualized.



Energy density profiles of three neutron stars with different central densities and masses. The low density stars do not contain any hyperons, whereas the other two stars do have hyperons in their inner core.



Time-time component of the metric tensor as a function of the radial coordinate. The solid line corresponds to the inner TOV-solution, whereas the dotted curve depicts the outer Schwarzschild part.

