Sheet 8

Hand in via OLAT until 19.01.2020 18:00.

22) Partition function of non-interacting particles *(4=2+2 Points)*

(i) Show that the partition function of a system of N non-interacting classical particles is related to the partition function of a single particle as follows:

$$Z_N = \frac{1}{N!} Z_1^N .$$

(ii) Show that in this case the chemical potential in the thermodynamic limit is:

$$\mu = -T \ln(Z_1/N).$$

23) Ideal gas in harmonic trap *(8=3+3+2 Points)*

Consider an ideales Gas in a harmonic trap. The Hamiltonian for each particle in the gas is given by

$$H = \frac{p_x^2 + p_y^2 + p_z^2}{2m} + \frac{1}{2} m\omega^2 (x^2 + y^2 + z^2).$$

The gas is in contact with a heat bath of temperature T and with a particle reservoir with chemical potential μ (i.e. the exchange of particles with the reservoir is allowed).

(i) Compute the grand canonical partition function of the system.

(ii) Consider the probability, that N particles are trapped. Show that this probability distribution is given by the Poisson distribution:

$$P(N) = \frac{\langle N \rangle^N}{N!} e^{-\langle N \rangle}.$$

(iii) Compute the average value of the total energy of the ideal gas $\langle E \rangle$ and the heat capacity

$$C_V = \left(\frac{\partial \langle E \rangle}{\partial T} \right)_{V, \langle N \rangle} .$$

24) Laplace transformation *(8=2+6 Points)*

In the lecture, the fugacity expansion was discussed, which allows to express the grand canonical partition function as a sum of canonical partition functions:

$$Z(T,V,\mu) = \sum_{N=0}^{\infty} Z(T,V,N) z^N, \qquad z = e^{\mu/T}.$$

Similarly, the canonical ensemble can be related to the microcanonical enseble, where instead of a power series in z the so-called Laplace transformation of a function $f(t)$ is used:

$$F(s) = \int_0^\infty dt e^{-st} f(t)$$

It is similar to a Fourier transformation, but the parameter s is not necessariy imaginary, and in the following it is assumed to be real.

(i) Show that the canonical partition function is the Laplace transform of the normalized microcanonical density of states:

$$Z(\beta,V,N) = \frac{1}{(2\pi\hbar)^{3N} N!} \int_0^\infty dE_0 e^{-E_0/T} \Omega(E_0,V,N).$$

Hint: introduce an idenity in integral representation.

(ii) Vice versa, an inverse Laplace transform of the canonical partition function can be defined, from which one can compute the microcanonical density of states:

$$\frac{1}{(2\pi\hbar)^{3N} N!} \Omega(E_0,V,N) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} d\beta e^{\beta E_0} Z(T,V,N), \qquad \beta = \frac{1}{T}.$$

Here, the integration path is parallel to the imatinary axis and the constant c has to be chosen such that it is larger than the largest real part of any singularity of the function $Z(T,V,N)$. Compute the microcanical densit of states for an ideal gas, given the canonical partition function $Z(T,V,N) = \frac{V^N}{N!}\left(\frac{mT}{2\pi\hbar^2}\right)^{3N/2}$.
Hint: Explain why you can put the integration path arbitrarily close to the imaginary axis. Close the integration path by a half circle at infinity and use the residue theorem: The contour integral of a function $f(z)$ in the complex plane z around a pole z_0 of degree p (i.e. the function f diverges as $1/(z-z_0)^p$) is given by:

$$\mathrm{Res}[f,z_0] = \frac{1}{2\pi i} \oint f(z) dz = \frac{1}{(p-1)!} \frac{d^{p-1}}{dz^{p-1}} \left[(z-z_0)^p f(z)\right]\Big|_{z=z_0}.$$