# Sheet 7

**Hand in via OLAT until 12.01.2020 18:00.**

## 18) N harmonic oscillators *(10=2+2+3+3 Points)*

The statistical distribution function for N harmonic oscillators of $N$ particles (with mass $m = 1$), with total energy $H = \frac{1}{2}\sum_{i=1}^{N}(p_i^2 + \omega^2 q_i^2)$ can be obtained from the Gibbs distribution function.

(i) Show that for the classical harmonic oscillator the spatial distribution function is as follows:

$$dw_q = \frac{\omega}{\sqrt{2\pi k_B T}} e^{-\frac{\omega^2 q^2}{2k_B T}} dq.$$

(ii) For $N$ quantum harmonical oscillators one yields:

$$dw_q = \left(\frac{\omega}{\pi\hbar}\tanh\frac{\hbar\omega}{2k_BT}\right)^{1/2} \exp\left(-q^2\frac{\omega}{\hbar}\tanh\frac{\hbar\omega}{2k_BT}\right) dq.$$

Show that in the classical limit $\hbar \to 0$, the classical result (i) is obtained.

(iii) What result is obtained in the limit $\hbar\omega \gg k_BT$ ? Show that this is the probability density of the position coordinate in the ground state.
*Hint: The wavefunction of a quantum mechanical harmonic oscillator in position space, with energy eigenvalues $E_n = \hbar\omega\left(n + \frac{1}{2}\right)$, is given as*

$$\psi_n(q) = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} H_n\left(q\sqrt{m\omega/\hbar}\right) e^{-\frac{m\omega}{2\hbar}q^2},$$

*with $H_n(x)$ the Hermite polynomials.*

(iv) Argue why the distribution function of the momentum coordinate $p$ is as follows:

$$dw_p = \left(\frac{1}{\pi\hbar\omega}\tanh\frac{\hbar\omega}{2k_BT}\right)^{1/2} \exp\left(-p^2\frac{1}{\hbar\omega}\tanh\frac{\hbar\omega}{2k_BT}\right) dp$$

and show, that in the limit $\hbar \to 0$ one obtains the Maxwell distribution.

**21) Ultrarelativistic ideal gas** ***(10=3+2+1+1+2+1 Points)***

Consider an ultrarelativistic ideal gas with $\epsilon_i = |\vec{p}_i|c$ the energy of the $i$-th particle in the ultrarelativistic limit.

(i) Derive the partition function $Z(T, V, N)$ in the canonical ensemble.

(ii) With this, compute the free energy $F(T, V, N)$ and make use of the Stirling formula to simplify.

(iii) Compute the pressure $P(T, V, N)$,

(iv) the chemical potential $\mu(T, V, N)$,

(v) the entropy $S(T, V, N)$ and

(vi) the internal energy $E(T, V, N)$.

**Christmas Exercise: Zipper Model**

*(Solving the Christmas exercise is voluntary. The best solution will be awarded a bottle of wine.)*

Consider a zipper with $N$ pieces. To open one piece the energy $\epsilon$ is necessary (a closed piece has energy 0). The zipper can only be opened thermally and from top to bottom. When opening the zipper, the piece with position $s$ can only be opened, if the pieces above $(1, 2, \ldots, s-1)$ are already open. We assume that every piece when opened is degenerated $g$ times.

(i) Calculate the canonical partition function as a function of $x = ge^{-\beta\epsilon}$, with $\beta = \frac{1}{T}$.

(ii) Compute with (i) the average number $\langle s \rangle$ of opened pieces as a function of $x$. Draw $\langle s \rangle (x)$ for $N = 1024$.

(iii) Compute the entropy and heat capacity at constant volume $C_V$. Investigate analytically the vicinity of $x = 1$ and consider the thermodynamic limit $N \to \infty$. Draw the quantities as a function of $x$, choosing $N = 1024$ and discuss the effect of choosing different degeneracy factors $g$.

(iv) Take a look at $\langle s \rangle (x)$ in the thermodynamic limit $N \to \infty$ now. Investigate the vicinity of $x = 1$. Is there a way to interpret the observed behavior physically?

(v) Calculate the temperature $T_c$ at $x = 1$ as a function of $\epsilon$ and $g$. Discuss the non-degenerate special case with $g = 1$.