Sheet 3

Hand in via OLAT until 24.11.2020 18:00.

7) The microcanonical ensemble in quantum statistics *(4+4=8 Points)*

In the lecture you derived the microcanonical ensemble in quantum statistics. For the density matrix of quantum mechanical states in a quasi-closed subsystem one has

$$\hat{\rho} = \sum_n \rho_n \hat{P}_n = \sum_n \rho_n \left|\psi_n\right\rangle \left\langle\psi_n\right| , \qquad \rho_n = \begin{cases} \frac{1}{\Delta\Gamma(E_0,V,N)} & E_0 \leq E_n \leq E_0 + \Delta E \\ 0 & \text{else} \end{cases} ,$$

with ψ_n an energy Eigenstate with Eigenvalue E_n.

(i) The general definition of a density matrix, familiar from the lecture is given as

$$\hat{\rho} = \sum_\alpha \omega_\alpha \hat{P}_\alpha = \sum_\alpha \omega_\alpha \left|\Psi_\alpha\right\rangle \left\langle\Psi_\alpha\right| ,$$

with probabilities $\sum_\alpha \omega_\alpha = 1$ and a set $\{\Psi_\alpha\}$ of pure states

$$\left|\Psi_\alpha\right\rangle = \sum_n c_{\alpha n} \left|\psi_n\right\rangle .$$

Show, that

$$\hat{\rho} = \sum_\alpha \omega_\alpha \hat{P}_\alpha = \sum_n \rho_n \hat{P}_n .$$

For this purpose, determine ρ_n as a function of ω_n and c_n, by evaluating $\mathrm{tr}\big(\hat{\rho}\big)$ in the energy basis.

(ii) Prove that

$$\langle \hat{H} \rangle = E_0,$$

is also true for the microcanonical ensemble in quantum statistics, in the case $\Delta E \ll 1$.

8) Equivalent definitions of the entropy *(3+3=6 Points)*

The volume of a d-dimensional sphere is given as

$$V(r) = \frac{2(2\pi)^{\frac{d-1}{2}}}{d!!} r^d,$$

with $d!!$ denoting the double faculty.

(i) Prove that for large dimensions d, nearly the complete volume of the sphere is located in a small layer close to the surface.
Hint: Consider the volume of a small spherical shell of thickness ΔR, that is located close to the surface of a sphere with radius R. Compare both volumes and consider the limit of large dimensions d.

(ii) In the lecture you defined the entropy of a system as

$$S(E_0, V, N) = \ln \Delta\Gamma(E_0, V, N),$$

with $\Delta\Gamma(E_0, V, N)$ denoting the phase space volume of a shell of thickness $[E_0 \leq E \leq E_0 + \Delta E]$. Use (i) to argue that the definitions

$$S(E_0, V, N) = \ln\big(\Gamma(E_0, V, N)\big), \qquad S(E_0, V, N) = \ln\big(\Omega(E_0, V, N)\big),$$

with $\Gamma(E_0, V, N)$ the phase space volume for $[E \leq E_0]$ and the density of states $\Omega(E_0, V, N)$, are equivalent in case of a statistical system.

9) Spin System *(2+4=6 Points)*

Consider a system of N free particles of spin $\frac{1}{2}$. The kinetic energy of the particles shall be small enough to be neglected. Every particle has a magnetic moment m because of its spin, which can be either parallel or anti-parallel with respect to an external field H. The energy of the system is then given by:

$$E_n = -(n - n')mH, \qquad n' = N - n,$$

with n the number of parallel (and n' antiparallel) spins.

(i) Compute the density of states $\Omega(E_n) = \frac{\Delta\Gamma(E_n)}{\Delta E}$ (with number $\Delta\Gamma(E_n)$ the number of microstates having energy E_n, and ΔE the minimal positive energy difference between mircostates).

(ii) Compute from this the entropy as a function of E_n. (Provide a sketch!)