

Sheet 10

Hand in via OLAT until 02.02.2020 18:00.

27) Adiabatic equations for an ideal gas (4=2+2 Points)

Consider an ideal gas with known heat capacity C_V and constant particle number N .

- (i) In the beginning the gas fills a volume V_1 with temperature T_1 and is compressed in an adiabatic way to a volume V_2 . Determine the work that has been performed in the process on one hand as a function of (C_V, γ, T_1) and on the other as a function of (C_V, γ, T_2) , with T_2 the temperature after the compression.
- (ii) Consider a circular process consisting of two isochoric processes and two adiabatic expansions following each other alternating. An ideal gas follows this circular process. Calculate the work performed.

28) Alternative derivation of the quantum statistics of ideal gases (6=2+2+2 Points)

We want to obtain the average occupation number of ideal gases. Consider an ideal gas, with the restriction, that the occupation numbers shall be limited to $n_k \in \{0, \dots, m\}$. Consider an ideal gas with energy levels ϵ_k for the atoms.

- (i) Compute the grand canonical partition function of the gas for the k -th quantum state Ω_k .

$$\text{Hint: } \sum_{k=0}^n q^k = \frac{1-q^{n+1}}{1-q}.$$

- (ii) Show that the average occupation number of this gas is given by

$$\langle n_k \rangle = \frac{1}{e^{\beta(\epsilon_k - \mu)} - 1} - \frac{m+1}{e^{\beta(\epsilon_k - \mu)(m+1)} - 1}.$$

- (iii) Compute $\langle n_k \rangle$ for the limiting cases $m \rightarrow \infty$ ($\mu < 0$) and $m = 1$ and identify with the known quantum gases.

26) Quantum gas in infinitely deep potential well (10=6+4 Points)

In the lecture the grandcanonical potential of a quantum gas of bosons or fermions has been discussed

$$\Omega(V, \mu, T) = -T \sum_k f\left(e^{-\beta(\epsilon_k - \mu)}\right),$$

with $f(x) = -\ln(1 - x)$ for bosons and $f(x) = \ln(1 + x)$ for fermions.

- (i) Consider an ideal quantum gas in a 3-dimensional infinitely deep square well ($-L/2 \leq x, y, z \leq L/2$).

Starting from the expression derived in the lecture, show that in the limit $V = L^3 \rightarrow \infty$, the grandcanonical potential of the gas is given as

$$\Omega(V, T, \mu) = -T \frac{V}{4\pi^2} \left(\frac{2m}{\hbar^2}\right)^{3/2} \int_0^\infty \sqrt{\epsilon} f(e^{-\beta(\epsilon - \mu)}) d\epsilon.$$

Use the boundary conditions of the wave functions of each particle

$$\psi(x = \pm L/2) = \psi(y = \pm L/2) = \psi(z = \pm L/2) = 0.$$

Hint: Use the energy eigenvalues of the quantum mechanical infinitely deep square well and the wave numbers \mathbf{k} .

- (ii) For relativistic particles the energy eigenvalues of the infinitely deep square well are given as

$$\epsilon(\mathbf{n}) = \sqrt{m^2 c^4 + \left(\frac{2\pi\hbar\mathbf{n}}{L}\right)^2},$$

with the momentum of a particle can be identified as $\mathbf{p} = \hbar\mathbf{k} = \frac{2\pi\hbar}{L}\mathbf{n}$, with $\mathbf{n} = (n_1, n_2, n_3)$ and $n_i \in \mathbb{Z}$.

Show in analogy to (i) that in the limit $L \rightarrow \infty$, the grandcanonical potential is given as

$$\Omega = -TV \int f\left(e^{-\beta(c\sqrt{m^2 c^2 + \mathbf{p}^2} - \mu)}\right) \frac{d^3 p}{(2\pi\hbar)^3}.$$