Exercise 1: Gluon and ghost propagator *(5+5+5=15 points)*

In the lecture the following generating functional of Yang Mills theory has been introduced

$$Z[J,\eta,\eta^\dagger] = e^{\mathrm{i}S_i\left[-\mathrm{i}\frac{\delta}{\delta J_\mu},-\mathrm{i}\frac{\delta}{\delta\eta},-\mathrm{i}\frac{\delta}{\delta\eta^\dagger}\right]} Z_0[J,\eta,\eta^\dagger], \tag{1}$$

with

$$\begin{aligned} Z_0[J,\eta,\eta^\dagger] = \frac{1}{Z_0}\int \exp\Big(&\mathrm{i}S_0[A_\mu,c,c^\dagger] \\ &+\mathrm{i}\int J^a_\mu(x)A^{\mu a}(x) + \eta^{\dagger a}(x)c^a(x) + \eta^a(x)c^{\dagger a}(x)d^4x\Big) DADc^\dagger Dc, \end{aligned} \tag{2}$$

with $SU(N)$ vector fields $A^{\mu a}$ and ghost fields c^a and $c^{\dagger a}$ from the gauge fixing procedure including the Fadeev Popov determinant. The free action of the theory is given as

$$S_0[A_\mu,c,c^\dagger] = \int\left(-\frac{1}{4}\left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu\right)^2 - \frac{1}{2\xi}\left(\partial_\mu A^{\mu a}\right)^2 + c^{\dagger a}\partial_\mu\partial^\mu c^a\right)d^4x. \tag{3}$$

i) Calculate the free generating functional of the ghost fields (you can neglect the gauge field parts), by solving the path integral, as you have done in the case of scalar field theory or Dirac fermions.
Hint: To solve the integral, calculate the Green's function of the ghost field operator, transform the integration variables and complete the square. The remaining integral need not be solved and cancels by normalization.

ii) Calculate the free generating functional $Z_0[J]$ of the SU(N) gauge fields by solving the path integral.
Hint: Repeat the same steps as in i) for the Yang-Mills fields. To solve the Green's function in momentum space, use the Ansatz

$$D^{\nu\ ab}_{\ \rho}(p) = \delta^{ab}\left[f_1(p)g^\nu_{\ \rho} + f_2(p)p^\nu p_\rho\right], \tag{4}$$

where $D^{\nu\ ab}_{\ \rho}(p)$ denotes the Yang-Mills propagator.

iii) Use your result to calculate the free gluon and ghost field two-point function

$$\langle 0|\,T\left(A^a_\mu(x)A^b_\nu(y)\right)|0\rangle = \left(-\mathrm{i}\frac{\delta}{\delta J^{\mu a}(x)}\right)\left(-\mathrm{i}\frac{\delta}{\delta J^{\nu b}(y)}\right)Z_0[J,\eta,\eta^\dagger]\Big|_{J,\eta,\eta^\dagger=0}, \tag{5}$$

$$\langle 0|\,T\left(c^a(x)c^{\dagger b}(y)\right)|0\rangle = \left(-\mathrm{i}\frac{\delta}{\delta \eta^{\dagger a}(x)}\right)\left(-\mathrm{i}\frac{\delta}{\delta \eta^{b}(y)}\right)Z_0[J,\eta,\eta^\dagger]\Big|_{J,\eta,\eta^\dagger=0}. \tag{6}$$

Exercise 2: 3-gluon vertex *(5 points)*

In this exercise we want to obtain the Feynman rule of the 3-gluon vertex, familiar from the lecture. The three gluon interaction of QCD is given by

$$\mathcal{L}_{int}[A_\mu] = -\frac{g}{2}\left(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a\right) f^{abc} A^{\mu b} A^{\nu c}. \tag{7}$$

Use the generating functional of the free gluons

$$Z_{0,YM}[J] = \exp\left(-\frac{1}{2}\int J^{\mu a}(x) D_{\mu\nu}^{ab}(x,y) J^{\nu b}(y) d^4x d^4y\right), \tag{8}$$

with the gluon propagator $D_{\mu\nu}^{ab}$, to calculate the 3-gluon vertex

$$V_3 = \mathrm{i}\int \mathcal{L}_{int}\left[-\mathrm{i}\frac{\delta}{\delta J_{\mu_i}}\right] d^4x Z_{0,YM}[J]. \tag{9}$$