Exercise 1: Fermion propagator *(4+2=6 points)*
Consider free Dirac fermions,

$$S_0[\bar{\psi}, \psi] = \int \bar{\psi} \left(\mathrm{i}\gamma^\mu \partial_\mu - m\right) \psi d^4x, \tag{1}$$

with generating functional

$$Z[\bar{\eta}, \eta] = \frac{1}{Z} \int e^{\mathrm{i}S_0[\bar{\psi},\psi]+\mathrm{i}\int\left(\bar{\eta}(x)\psi(x)+\bar{\psi}(x)\eta(x)\right)d^4x} D\psi D\bar{\psi}. \tag{2}$$

i) Explicitly solve the integral and show, that the generating functional can be given as

$$Z[\bar{\eta}, \eta] = e^{-\int \bar{\eta}(x) S_F(x,y) \eta(y) d^4x d^4y}, \tag{3}$$

with

$$S_F(x, y) = \int \frac{\mathrm{i}}{\not{k} - m + \mathrm{i}\epsilon} e^{-\mathrm{i}k(x-y)} \frac{d^4k}{(2\pi)^4}. \tag{4}$$

ii) Show, that S_F is indeed the Feynman propagator, by calculating

$$\langle 0| T\left(\psi(x)\bar{\psi}(y)\right)|0\rangle = \left(-\mathrm{i}\frac{\delta}{\delta\bar{\eta}(x)}\right)\left(\mathrm{i}\frac{\delta}{\delta\eta(y)}\right) Z[\bar{\eta}, \eta]\Big|_{\bar{\eta}=\eta=0}. \tag{5}$$

Exercise 2: Yukawa theory *(2+4=6 points)*
Consider the following Lagrangian for a theory with interacting real scalar fields ϕ and Dirac fermion fields ψ, referred to as a Yukawa theory

$$\mathcal{L}[\phi, \psi, \bar{\psi}] = \frac{1}{2}\partial^\mu\phi\partial_\mu\phi + \bar{\psi}\left(\mathrm{i}\not{\partial} - m_\psi\right)\psi - \frac{1}{2}m_\phi^2\phi^2 + g\bar{\psi}\psi\phi. \tag{6}$$

i) Give the generating functional of the theory $Z[J, \eta, \bar{\eta}]$ and express it as a function of the generating functional of the non-interacting theory $Z_0[J, \eta, \bar{\eta}]$, as you have done it in the lecture for scalar ϕ^4 theory.

ii) Calculate the connected Green's function to order $\mathcal{O}(g^2)$

$$\langle\Omega| T\left(\psi(x)\bar{\psi}(y)\right)|\Omega\rangle, \tag{7}$$

using the generating functional.

Exercise 3: Fadeev-Popov method *(2+4+2=8 points)*
Let us solve the 2-dim. Gaussian integral

$$I = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2} dxdy, \tag{8}$$

in a Fadeev-Popov inspired way.

i) Argue, that the integral is invariant under rotation

$$\begin{pmatrix} x^g \\ y^g \end{pmatrix} = g \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{9}$$

where g is a $SO(2)$ matrix.

ii) Use the relation

$$1 = \Delta_{FP}(x,y) \int \delta\left(F(x^g, y^g)\right) dg, \tag{10}$$

to calculate $\Delta_{FP}^{-1}(x,y)$, using the following „gauge“

$$F(x^g, y^g) = y^g = x\sin\phi + y\cos\phi. \tag{11}$$

Hint: Use the following relation to simplify the δ-function

$$\delta(f(x)) = \sum_i \frac{\delta(x-x_i)}{|f'(x_i)|}, \tag{12}$$

where x_i are the roots of $f(x_i) = 0$.

iii) Use your results to show that

$$I = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2} dxdy = \int dg \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Delta_{FP}(x,y)\delta\left(F(x,y)=y\right) e^{-x^2-y^2} dxdy = \pi, \tag{13}$$

where $\int dg$ is the volume of the gauge orbit.