Exercise 1: Connected functions *(2+3+3=8 points)*
Consider an interacting, scalar quantum field theory with generating functional $Z[J]$. Using the generating functional for the connected Green's functions $W[J] = \log Z[J]$, show the following relations ($\langle ... \rangle = \langle \Omega | T(...) | \Omega \rangle$)

i) For the one-point function

$$\langle \phi(x_1) \rangle = \langle \phi(x_1) \rangle_c = \langle \phi \rangle . \tag{1}$$

ii) For the two-point function

$$\langle \phi(x_1)\phi(x_2) \rangle = \langle \phi(x_1)\phi(x_2) \rangle_c + \langle \phi \rangle^2 . \tag{2}$$

iii) For the three-point function

$$\begin{aligned} \langle \phi(x_1)\phi(x_2)\phi(x_3) \rangle =& \langle \phi(x_1)\phi(x_2)\phi(x_3) \rangle_c \\ &+ \langle \phi \rangle \left(\langle \phi(x_1)\phi(x_2) \rangle_c + \langle \phi(x_2)\phi(x_3) \rangle_c + \langle \phi(x_1)\phi(x_3) \rangle_c \right) + \langle \phi \rangle^3 . \end{aligned} \tag{3}$$

Exercise 2: Dyson-Schwinger equation *(4+4=8 points)*
Consider an euclidean, real scalar field theory with a $\mathcal{L}_{\text{int}}[\phi] = \frac{\lambda}{4!}\phi^4$ interaction. In the lecture you derived the Dyson-Schwinger equation for a scalar field theory with generating functional $Z[J]$,

$$\left(-\frac{\delta S[\phi]}{\delta \phi(x)} \left[\frac{\delta}{\delta J} \right] + J(x) \right) Z[J] = 0. \tag{4}$$

Use this expression to derive exact equations that are satisfied

i) by the two- and four-point (with three points identified) functions.

ii) by the four- and six-point (with three points identified) functions.

Exercise 3: Connected Green's functions and the effective action *(4 points)*
In the lecture you have found the following relations between amputated Green's functions G_n^a and the proper vertices $\Gamma^{(n)}$, for a general action $S[\phi]$

$$\begin{aligned} \Gamma^{(1)}(x) =& 0 \\ \Gamma^{(2)}(x_1, x_2) =& S(x_1, x_2) = [G_2^c(x_1, x_2)]^{-1} \\ \Gamma^{(3)}(x_1, x_2, x_3) =& -G_3^a(x_1, x_2, x_3) \\ \Gamma^{(4)}(x_1, x_2, x_3, x_4) =& -G_4^a(x_1, x_2, x_3, x_4) + \int G_3^a(x_1, x_2, y) G_2^c(y, z) G_3^a(x_3, x_4, z) d^4y d^4z \\ &+ 2 \text{ permutations.} \end{aligned} \tag{5}$$

Identify the permutations and use the relations to give the connected Green's functions

$$G_3^c(x_1, x_2, x_3) \quad \text{and} \quad G_4^c(x_1, x_2, x_3, x_4), \tag{6}$$

as functions of $\Gamma^{(i)}$.
What happens in the special case of ϕ^4-theory?