Exercise 1: Vacuum expectation value for the harmonic oscillator *(4 points)*
In the lecture you found the following euclidean generating functional for the one-dimensional harmonic oscillator

$$Z_E[j] = \exp\left(\frac{1}{4m\omega}\int\int j(\tau_1)e^{-\omega|\tau_1-\tau_2|}j(\tau_2)d\tau_1 d\tau_2\right). \tag{1}$$

Calculate the vacuum expectation value of

$$\langle 0| x^2 |0\rangle , \tag{2}$$

using $Z_E[j]$.

Exercise 2: Euclidean path integral *(6 points)*
Following the same steps as in the lecture, derive the one dimensional, euclidean path integral

$$\langle x_2| U(\tau_2-\tau_1) |x_1\rangle = \int_{y(\tau_1)=x_1}^{y(\tau_2)=x_2} e^{-S_E[y]}Dy, \tag{3}$$

with evolution operator (for imaginary time $\tau_2 > \tau_1$)

$$U(\tau_2-\tau_1) = e^{-(\tau_2-\tau_1)H}, \tag{4}$$

with a time independent Hamilton operator $H = H_0 + V = \frac{p^2}{2m} + V$. The quantity $S_E[y]$ is the euclidean action

$$S_E[y] = \int\left(\frac{m\dot{y}^2}{2} + V\right)d\tau, \tag{5}$$

that needs to be identified in the derivation.

Exercise 3: Gaussian integrals II *(5+5=10 points)*
Prove the following Gaussian integrals

i) Let $x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$ be a n-dimensional vector and $A \in \mathbb{R}^{n\times n}$ be a real, symmetric matrix, with positive eigenvalues $\lambda_i > 0$, that can be diagonalized. Prove that

$$\int e^{-\frac{1}{2}x^T Ax}d^n x = \sqrt{\frac{(2\pi)^n}{\det(A)}}. \tag{6}$$

Hint: Diagonalize the matrix A (recall the spectral theorem).

ii) In the same manner prove that

$$\int e^{-\frac{1}{2}x^T Ax + J^T x}d^n x = \sqrt{\frac{(2\pi)^n}{\det(A)}}e^{\frac{1}{2}J^T A^{-1}J}. \tag{7}$$