Exercise 1: Gaussian integrals *(2+2=4 points)*
Calculate the following Gaussian integrals

i) Calculate

$$\langle x^{2n} \rangle = \sqrt{\frac{a}{2\pi}} \int_{-\infty}^{\infty} x^{2n} e^{-\frac{1}{2}ax^2} dx \tag{1}$$

ii) Calculate

$$\langle x^{2n} \rangle = \left(\frac{d}{dJ}\right)^{2n} \left[\sqrt{\frac{a}{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}ax^2 + Jx} dx \right] \Bigg|_{J=0} \tag{2}$$

Exercise 2: Functional derivative - chain rule*(2+2=4 points)*
Recall the definition of the functional derivative

$$\delta F[x] = \int \frac{\delta F[x]}{\delta x(s)} \delta x(s) ds. \tag{3}$$

Consider the following functional

$$F[g(f)] = \int g(f(x))dx, \quad \text{with} \quad g(f) = (f'(x))^n \,. \tag{4}$$

i) Calculate the derivative using

$$\frac{\delta F[g(f)]}{\delta f(y)} = \int \frac{\delta F[g(f)]}{\delta [g(f)](s))} \frac{\delta [g(f)](s)}{\delta f(y)} ds. \tag{5}$$

ii) Calculate the derivative using

$$\frac{\delta F[g(f)]}{\delta f(y)} = \frac{d}{ds} F[g(f(x) + s\delta(x-y))]\Big|_{s=0}. \tag{6}$$

Exercise 3: Partition function of the harmonic oscillator *(3+3+3+3=12 points)*
Recall the Hamilton function of the 1 dimensional harmonic oscillator

$$H(p,x) = T + V = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2. \tag{7}$$

i) Use the Hamilton function to compute the partition function of the system

$$Z_1 = \int e^{-\beta H}\frac{dpdx}{2\pi}, \tag{8}$$

with $\beta = 1/T$.

ii) We can obtain a similar result from the quantum mechanical case. Recall the energies of the discrete spectrum of the quanum mechanical harmonic oscillator E_n and calculate

$$Z_2 = \sum_n e^{-\beta E_n}. \tag{9}$$

iii) Finally another possibility to obtain the partition function is using the Mehler formula given by

$$\langle x| e^{-\beta H} |x\rangle = \sqrt{\frac{m\omega}{2\pi\sinh(\beta\omega)}}\exp\left(-\frac{x^2 m\omega}{\sinh(\beta\omega)}(\cosh(\omega\beta) - 1)\right). \tag{10}$$

Calculate the partition function using

$$Z_3 = \int \langle x| e^{-\beta H} |x\rangle\, dx. \tag{11}$$

iv) Calculate the free energy

$$F = -\frac{1}{\beta}\log(Z), \tag{12}$$

for all three cases Z_1, Z_2 and Z_3 and discuss the limit $\beta << 1$ for F_2 and F_3.