Exercise 1: Zassenhaus Formula *(5+5=10 points)*
Consider two operators A and B, with $[A,[A,B]] = [B,[A,B]] = 0$.

i) Derive the Baker-Campbell-Hausdorff formula

$$e^{A+B} = e^A e^B e^{-\frac{1}{2}[A,B]}. \tag{1}$$

For this purpose show

$$[A^n, B] = nA^{n-1}[A,B] \tag{2}$$

first and then derive the following differential equation by definining $f(t) = e^{tA}e^{tB}$

$$\frac{df}{dt} = (A + B + t[A,B])f(t). \tag{3}$$

Find another solution of this differential equation.
Hint: Calculating the commutator $[e^{tA}, B]$ *will be of great use.*

ii) If the relations $[A,[A,B]] = [B,[A,B]] = 0$ do not hold, the formula can be extended to general cases. This extension is known as Zassenhaus formula and given by

$$e^{A+B} = e^A e^B e^{-\frac{1}{2}[A,B]} e^{-\frac{1}{6}(2[B,[B,A]]+[A,[B,A]])} e^{Z_3} e^{Z_4} ..., \tag{4}$$

where $Z_3, Z_4, ...$ contain higher powers of the operators A and B. Use this formula to show the following relation for the Hamilton operator $H = H_0 + V$

$$e^{-\mathrm{i}\epsilon H} = e^{-\mathrm{i}\epsilon(H_0+V)} = e^{-\mathrm{i}\epsilon V/2} e^{-\mathrm{i}\epsilon H_0} e^{-\mathrm{i}\epsilon V/2} + \mathcal{O}(\epsilon^3), \tag{5}$$

with small parameter $\epsilon > 0$.