Exercise 1: Differential cross section

In the lecture you have discussed the differential cross section of two incoming particles with four-momenta p_1 and p_2. The differential cross section for 2-2 scattering (two incoming and two outgoing particles) is given as

$$d\sigma = \frac{1}{4\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}} (2\pi)^4 \delta^{(4)}(k_1 + k_2 - p_1 - p_2) |M_f|^2 \frac{d^3 k_1}{(2\pi)^3 2k_1^0} \frac{d^3 k_2}{(2\pi)^3 2k_2^0}, \tag{1}$$

Let us consider the incoming particles would carry the same mass m and the outgoing ones the mass M. Show that the obtained cross section of 2-2 scattering leads to

$$\frac{d\sigma}{d\Omega} = \frac{\sqrt{1 - 4\frac{M^2}{s}}}{64\pi^2 s \sqrt{1 - 4\frac{m^2}{s}}} |M_{f_i}|^2, \tag{2}$$

in the center-of-mass system.
Hint: We define the total energy of the collision in the center-of-mass system $E_{tot} = \sqrt{s}$.

Exercise 2: Wick's theorem

In the lecture you showed Wick's theorem for the time ordered product of two field operators

$$T\left(\hat{\phi}_I(x_1)\hat{\phi}_I(x_2)\right) =: \hat{\phi}_I(x_1)\hat{\phi}_I(x_2) : + \langle 0| T\left(\hat{\phi}_I(x_1)\hat{\phi}_I(x_2)\right)|0\rangle . \tag{3}$$

Use this result to prove that

$$\begin{aligned} T\left(\hat{\phi}_I(x_1)\hat{\phi}_I(x_2)\hat{\phi}_I(x_3)\right) =&: \hat{\phi}_I(x_1)\hat{\phi}_I(x_2)\hat{\phi}_I(x_3) : + : \hat{\phi}_I(x_1) \langle 0| T\left(\hat{\phi}_I(x_2)\hat{\phi}_I(x_3)\right)|0\rangle \\ &+ : \hat{\phi}_I(x_2) \langle 0| T\left(\hat{\phi}_I(x_1)\hat{\phi}_I(x_3)\right)|0\rangle + : \hat{\phi}_I(x_3) \langle 0| T\left(\hat{\phi}_I(x_1)\hat{\phi}_I(x_2)\right)|0\rangle \end{aligned} \tag{4}$$

Hint: You will need to decompose the fields at some point, as you have done it in the lecture, writing $\hat{\phi}_I = \hat{\phi}_I^+ + \hat{\phi}_I^-$. Also show it for $x_1^0 \geq x_2^0 \geq x_3^0$ at first and discuss afterwards that it holds in all cases.

Exercise 3: Feynman propagator of real scalar field theory

In the lecture you calculated the Feynman propagator of real scalar field theory

$$\Delta_F(x-y) = \mathrm{i}\int \frac{e^{-\mathrm{i}p(x-y)}}{p^2 - m_0^2 + \mathrm{i}\epsilon}\bigg|_{\epsilon=0} \frac{d^4p}{(2\pi)^4} \tag{5}$$

i) Prove that the propagator is given as

$$\Delta_F(x-y) = \int \left(e^{-\mathrm{i}p(x-y)}\theta(t_x - t_y) + e^{\mathrm{i}p(x-y)}\theta(t_y - t_x)\right) \frac{d^3p}{(2\pi)^3 2E(\mathbf{p})}, \tag{6}$$

when performing part of the integration.
Hint: Find the poles of the denominator and use the residue theorem. You will have to rescale $\epsilon' = \frac{\epsilon}{2E(\mathbf{p})}$.

ii) We want to show that the Feynman propagator is the Green function of the Klein-Gordon operator

$$(\partial_\mu \partial^\mu + m_0^2)\Delta_F(x-y) = (\partial_\mu \partial^\mu + m_0^2)\langle 0| T\left(\hat{\phi}_I(x)\hat{\phi}_I(y)\right)|0\rangle = -\mathrm{i}\delta^{(4)}(x-y). \tag{7}$$

Prove that this is indeed the case by letting the operator $(\partial_\mu \partial^\mu + m_0^2)$ act on expectation value of the time ordered product.
Hint: You will need the derivative of the delta distribution, that is only defined in a distributional sense as

$$\left(\frac{\partial}{\partial t_x}\delta(t_x - t_y)\right)\hat{\phi}_I(x) = -\delta(t_x - t_y)\frac{\partial}{\partial t_x}\hat{\phi}_I(x). \tag{8}$$